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POLYNOMIAL BOUNDS FOR BIUNIVALENT FUNCTION ASSOCIATED WITH THE PROBABILITY OF GENERALIZED DISTRIBUTION DEFINED BY GENERALIZED POLYLOGARITMS VIA CHEBSHEV POLYNOMIAL

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Abstract
We introduced and investigated new subclasses $M_{\Sigma}^{m}(\lambda,\alpha,S,\phi(z,t))$ and $\Omega_{\Sigma}^{m}(\beta,\alpha,S,\phi(z,t))$ of bi-univalent functions defined in the open unit disk, involving the probabilities of generalized distribution and generalized polylogarithms subordinate to Chebyshev polynomials. Furthermore, we determined the estimates on the Taylor-Maclaurin coefficients $\left|\frac{a_1}{S}\right|$ and $\left|\frac{a_2}{S}\right|$ for the functions in the new subclasses introduced here. Also, we used these estimates to established the relevant connections to classical Fekete-Szego estimate. Several presume new consequences of the results were also established.
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Introduction

Let A represents the class of function $F(Z)$ analytic in the open unit disk E = {z: |z| < 1} which has the Taylor series expansion of the form

$$f(z) = z + a_w Z^2 + a_3 Z^3 + \ldots, \qquad (1)$$

and also normalized with $f(0) = f'(0) - 1 = 0$. Denote S to be the subclass of A consisting of univalent functions in E. Subsequently, denote S^* and C to be subclasses of S which are starlike and convex, respectively, in the unit disk E. For formal definitions (including analytical) of the starlike and convex functions and other insights on them, refer to (Duren, 1983 and Goodman, 1983).

By the Koebe one-quarter theorem in Duren (1983), we know that the image of E under every univalent function $f \in A$ contains the disk with center in the origin and radius ¼. Therefore, every univalent function $f \in A$ has an inverse f^1 defined by

$$f^{-1}(f(z)) = z \qquad (z \in E)$$
$$f\left(f^{-1}(w)\right) = w \qquad \left(|w| < r_o(f) : r_o(f) \geq \tfrac{1}{4}\right). \qquad (2)$$

It is easy to see that the inverse function has the form:

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^3 - 5a_2 a_3 + ay\right)w^4 + \ldots \qquad (3)$$

A function $f \in A$ is said to be bi-univalent in E if both $f(z)$ and $f^1(z)$ are univalent in E and is usually denoted by Σ.

Literature shows that Lewin (1967) started the class of bi-univalent functions and he actually proved that the modulus of second Maclaurin Taylor coefficient $|a_2| < 1.51$. Thereafter, Brannan and Clunie (1970) and Netanyahu (1969) asserted that modulus of a_2 is less or equal to $\sqrt{2}$ and 4/3 respectively. For a brief account and absorbing examples of functions in the class Σ can be found in the breaking new ground work of Srivastava et al. (2013), which has evidently renewed the study of bi-univalent functions while ago. Actually, since the work of Srivastava et al. (2013), a colossal overfill of papers have surfaced and are still surfacing in the literature dealing with various subclasses of the bi-univalent and other related function classes. For more on bi-univalent, see (Awolere and Oladipo, 2019; Guney et al., 2017; Li et al., 2015; Murugusundaramoorthy and Janani, 2015; Murugusundaramoorthy, 2015; Porwal, 2018; Laxmi and Sharma, 2017; Srivastava et al., 2010 and Srivastava et al., 2015).

Chebyshev polynomials play vital role in numerical analysis. In literature the most books and research articles related to specific orthogonal polynomials of Chebyshev family contain essentially results of Chebyshev polynomials of first and second kinds in *log* and *Un*(*x*) and their very many uses in different applications, see (Doha, 1994; Awolere and Oladipo, 2019).

It is well-known that the most kinds of Chebyshev polynomials are first and second kinds and in case of variable *x* on (-1, 1), the first and second kinds are defined by

$$Tn(x) = \cos n\theta \tag{4}$$

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \tag{5}$$

where n denotes the degree of the polynomial and x = cos θ. Chebyshev polynomials of the second kind have the generating function of the form

$$H(z,t) = \frac{1}{1-2xz+z^2} = 1 + \sum_{k=1}^{\infty} \frac{\sin(n+1)\theta}{\sin\theta} z^n. \tag{6}$$

We note that if $t = \cos\theta, \theta \in \left(-\pi/3, \pi/3\right)$, then

$$H(z,t) = \frac{1}{1-2\cos\theta\, z + z^2} = 1 + \sum_{k=1}^{\infty} \frac{\sin(k+1)\theta}{\sin\theta} z^k$$

$$= 1 + U_1(t)z + U_2(t)z^2 + \ldots \quad (z \in E, t \in (-1,1))$$

where

$$U_{k-1} = \frac{\sin(k\ arc\ \cos t)}{\sqrt{1-t^2}}, k \in N \tag{7}$$

is the Chebyshev polynomial of the second kind. We also noted that

$$U_k(t) = 2t\, U_{k-1}(t) - U_{k-2}(t) \tag{8}$$

So that

$$U_{1k}(t) = 2t, U_2(t) = 4t^2 - 1, U_3(t) = 8t^3\ 4t \tag{9}$$

Very recently, Porwal (2018) introduced and studied generalized distribution and its geometric properties associated with univalent functions which he represented by

$$K_\varphi(z) = z + \sum_{k=2}^{\infty} \frac{a_{k-1}}{S} z^k, \tag{10}$$

where $S = \sum_{k=0}^{\infty} a_k, a_k \geq 0$. Other related work on function of the form (10) can be found in Oladipo (2019a).

Also, Al-Shaqsi and Darus (2008) generalized Rusheweyh and Salagean operators as

$$D_\lambda^m f(z) = z + \sum_{k=2}^{\infty} \frac{k^m (k+\lambda-1)1}{\lambda!(k-1)1} z^k, \tag{11}$$

where $(m \in N) = \{0,1,2,\ldots\}$, $z \in E$. We note that the derivative operator $D_\lambda^m f(z)$ incorporated two operators. If λ = 0, the operator reduces to Salagean differential operator and if $m = 0$ the operator will reduce to Rusheweyh operator.

Now, we recall from Fadipe et al., (2018) the function

$$f_\gamma(z) = z + \sum_{k=2}^{\infty} \gamma(s) a_k z^k, \tag{12}$$

where $\gamma(s) = \frac{2}{1+e^{-s}}$, $s \geq 0$ and s is real is the modified sigmoid function. Functions of the form (12) belong the Class A where $A_1 = A$.

By (4) and (6) we let

$$K_{\psi\gamma}(z) = z + \sum_{k=2}^{\infty} \gamma(s) \frac{a_{k-1}}{s} z^k . \quad (13)$$

By (11) and (13) using convolution or (Hadamard principle) we defined

$$D^m f_\gamma(z) = z + \sum_{k=2}^{\infty} \frac{k^m (k+\theta-1)!}{\lambda!(k-1)!} \gamma(s) \frac{a_{k-1}}{s} z^k, \quad (14)$$

where $\theta, \lambda \in N\, U\{0\}$ and $s = \sum_{k=0}^{\infty} a_k,\ \gamma(s) = \frac{2}{1+e^{-s}},\ s \geq 0$ and s is real. For more insight on these see (Fadipe-Joseph et al., 2013; Gbolagade et al., 2014).

Further for functions g of the form (3), we defined

$$D_\lambda^m g_\gamma(w) = w - \frac{a_1}{s} A_2 w^2 + A_3\left(2\frac{a_1^2}{s^2} - \frac{a_2}{s}\right) w^3 + \ldots, \quad (15)$$

where

$A_k = \frac{k^m (k+\lambda-1)!}{\lambda!(k-1)!} \gamma(s)$ such that $A_2 = (1+\theta)2^m$

$$A_3 = \frac{3^m}{2}(1+\theta)(2+\theta), \quad (16)$$

and we consequently introduced two new subclasses of bi-univalent functions.

Motivated by recent works of Guney et al., (2017) on bi-univalent functions involving Salagean operator and that of Altinkaya and Yalcin (2016), we introduced two new subclasses of Σ associated with Chebyshev polynomial whose polynomial are coefficients of probabilities of the generalized distribution and obtain the initial Taylor coefficients $\left|\frac{a_1}{S}\right|$ and $\left|\frac{a_2}{S}\right|$ for the function classes defined via subordination principle.

Definitions and Lemmas

Definition 1: For 0≤ λ ≤ 1 and $t(-1,1)$ a function $f \in \Sigma$ of form (1) is said to be in the class $M_\Sigma^m(\lambda, \theta, \gamma(s), \phi(z,t))$ if the following subordination holds:

$$(1-\lambda)\frac{D_\lambda^{m+1} f_\gamma(z)}{D_\lambda^m f_\gamma(z)} + \lambda \frac{D_\lambda^{m+2} f_\gamma(z)}{D_\lambda^{m+1} f_\gamma(z)} \prec \varphi(z,t) \quad (17)$$

$$(1-\lambda)\frac{D_\lambda^{m+1} g_\gamma(w)}{D_\lambda^m g_\gamma(w)} + \lambda \frac{D_\lambda^{m+2} g_\gamma(w)}{D_\lambda^{m+1} g_\gamma(w)} \prec \varphi(w,t), \quad (18)$$

where $z, w \in \Sigma$ and g is given by (3).

We observe that by specializing the parameters λ, θ, S and suitably fixing the values of m in Definition 1, we presumably introduce the following new subclasses of Σ as listed below.

Remark 2: Supposing $f(z) \in \Sigma$ and $f \in (-1,1)$, then we represent

$$\begin{aligned}
&(1) \quad M_\Sigma^m(\phi, \theta, \gamma(s), \phi(z,t)) = S_\Sigma^m(\theta,, \phi(z,t)\gamma(s)) \\
&(2) \quad M_\Sigma^m(1, \theta, \gamma(s), \phi(z,t)) = C_\Sigma^m(\theta,, \phi(z,t)\gamma(s)) \\
&(3) \quad M_\Sigma^o(0, \theta,, \phi(z,t), \gamma(s)) = S_\Sigma^m(\theta,, \phi(z,t)\gamma(s)) \\
&(4) \quad M_\Sigma^o(1, \theta,, \phi(z,t), \gamma(s)) = C_\Sigma^m(\theta,, \phi(z,t)\gamma(s))
\end{aligned} \quad (19)$$

due to (Guney, et al., 207).

We defined the following new subclasses involving generalized polylogarithm in Oladipo (2019b) and probabilities of generalized distribution in (Porwal, 2018).

Definition 3: For α ≤ β ≤ 1 and $t \in (-1,1)$ a function $f \in \Sigma$ of the form (1) is said to be in the class $\alpha_\Sigma^m(\lambda, \theta, \gamma(s), \phi(z,t))$ if the following subordination holds:

$$(1-\beta)\frac{D_\lambda^m f_\gamma(z)}{z} + \beta\left(D_\lambda^m f_\gamma(z)\right)' \prec \varphi(z,t) \quad (20)$$

$$(1-\beta)\frac{D_\lambda^m g_\gamma(z)}{w} + \beta\left(D_\lambda^m g_\gamma(w)\right)' \prec \varphi(w,t) \quad (21)$$

Remark 4: Let $f(z) \in \Sigma$ and $f \in (-1,1)$, then we represent

$$\begin{aligned}
&(1) \quad F_\Sigma^m(0, \theta, \gamma(s), \phi(z,t)) = B_\Sigma^m(\theta, \gamma(s), \phi(z,t)) \\
&(2) \quad F_\Sigma^m(1, \theta, \gamma(s), \phi(z,t)) = K_\Sigma^m(\theta, \gamma(s), \phi(z,t)) \\
&(3) \quad F_\Sigma^o(\beta, \theta,, \gamma(s), \phi(z,t)) = S_\Sigma^m(\beta, \theta, \gamma(s), \phi(z,t)) \\
&(4) \quad F_\Sigma^o(1, \theta, \gamma(s), \phi(z,t)) = R_\Sigma^o(\theta, \gamma(s), \phi(z,t))
\end{aligned} \quad (22)$$

Theorem 1: Let $f \in M_\Sigma^m(\lambda, \theta, S, \gamma(s), \phi(z,t))$. then

$$\left|\frac{a_1}{s}\right| \leq \frac{2t\sqrt{2t}}{\sqrt{\left[(1+2\lambda)(\theta+1)(\theta+2)3^m \gamma(s) - (\lambda^2+5\lambda+2)2_{\gamma(s)}^{2m}(\theta+1)\right]4t^2 + 2m(1+\lambda)^2(\theta+1)^2\gamma^2(s)}}$$

$$\left|\frac{a_2}{s}\right| \leq \left(\frac{2t}{2^m(1+\lambda)(1+\theta)\gamma(s)}\right)^2 + \frac{2t}{3^m(1+2\lambda)(1+\theta)(\theta+2)\gamma(s)}$$

Proof: From Definitions 1 and 3, we have

$$(1-\lambda)\frac{D^{n+1} f_\gamma(z)}{D^m f_\gamma(z)} + \lambda \frac{D^{n+2} f_\gamma(z)}{D^{m+2} f_\gamma(z)} = U_1(t)U(z) + U_2(t)U^2(z) + \ldots. \quad (25)$$

$$(1-\lambda)\frac{D^{m+1} g_\gamma(Z)}{D^{m+1} g_\gamma(w)} + \lambda \frac{D^{m+2} g_\gamma(w)}{D^{m+2} g_\gamma(w)} = 1 + U_1(t)V(w) + U_2(t)V^2(w) + \ldots. \quad (26)$$

In the light of (25) and (26), we have

$$1 + (1+\lambda)A_2 \frac{a_1}{s} z + \left[2(1+2\lambda)A_3 \frac{a_2}{s} - (1+3\lambda)A_2^2 \frac{a_1^2}{s^2}\right] z^2$$

$$+ \ldots = 1 + U(t)C_1 z + \left[U_1(t)C_2 + U_2(t)C_1^2\right] z^2 \quad (27)$$

and

$$1-(1+\lambda)A_2\frac{a_1}{s}w+\left\{[(8\lambda+4)A_3-(3\lambda+1)A_2]\frac{a_1^2}{s^2}-2(1+2\lambda)A_3\frac{a_1^2}{s^2}\right\}w^2$$

$$+\ldots=1+U_1(t)C_1w+\left[U_1(t)C_2+U_2(t)C_1^2\right]w^2+\ldots \quad (28)$$

Thus, the following relation was obtained by equating the power of $z's$

$$(1+\lambda)A_2\frac{a_1}{s}=U_1(t)C_1 \quad (29)$$

$$2(1+2\lambda)A_3\frac{a_2}{s}-(1+3\lambda)A_2^2\frac{a_1^2}{s^2}=U_1(t)C_2+U_2(t)C_1^2 \quad (30)$$

$$(1+\lambda)A_2\frac{a_1}{s}=U_1(t)d_1 \quad (31)$$

$$1-(1+\lambda)A_2\frac{a_1}{s}+\left\{[(8\lambda+4)A_3-(3\lambda+1)A_2]\frac{a_1^2}{s^2}-2(1+2\lambda)A_3\frac{a^2}{2}\right\}=U_1(t)d_2+U_2(t)d_1^2 \quad (32)$$

$$C_1=-d_1 \quad (33)$$

$$2(1+\lambda)^2A_2^2\frac{a_1^2}{s^2}=U_1^2(t)\left[C_1^2+d_1^2\right] \quad (34)$$

Adding (30) to (32) and using (33) and (34), we obtain

$$\frac{a_1^2}{s^2}=\frac{U_1^3(t)[C_2+d_2]}{2\left[\{2(1+2\lambda)A_3-(1+3\lambda)A_2^2\}U_1^2(t)-(1+\lambda)^2A_2^2U_2(t)\right]} \quad (35)$$

By applying Lemma 1 to the coefficients C_2 and d_2 and making use of (9) we have

$$\left|\frac{a_1}{s}\right|\le\frac{U_1^3(t)[C_2+d_2]}{\sqrt{\left|\left[(1+2\lambda)(\theta+1)(\theta+2)3^m\gamma(s)-(\lambda^2+5\lambda+2)2^{2m}_{\gamma(s)}(\theta+1)\right]4t^2+2m(1+\lambda)^2(\theta+1)^2\gamma^2(s)\right|}}$$

By subtracting (32) from (30) and using (33) and (34), we get

$$\frac{a_2}{s}=\frac{U_1^2(t)\left[C_1^2+d_1^2\right]}{2^{2mt}(1+\lambda)^2(1+\theta)^2}+\frac{U_1(t)[C_2-d_2]}{2(1+2\lambda)(1+\theta)(\theta+2)3^m} \quad (36)$$

Applying Lemma 1 once again to the coefficients together with (9), we get

$$\left|\frac{a_2}{s}\right|\le\left(\frac{2t}{2^m(1+\lambda)(1+\theta)\gamma(s)}\right)^2+\frac{2t}{3^m(1+2\lambda)(1+\theta)(\theta+2)\gamma(s)}$$

By taking λ = 0 or λ = 1 and $t\in(0,1)$, one can easily state the estimates $\left|\frac{a_1}{s}\right|$ and $\left|\frac{a_2}{s}\right|$ for the function classes M_Σ^n

Remark 6: Let $f\in M_\Sigma^m(m,\lambda,\theta,\gamma(s),\phi(z,t))$

$$\left|\frac{a_1}{s}\right|\le\frac{2t\sqrt{2t}}{\sqrt{\left|[(1+\theta)(\theta+2)3^m\gamma(s)-2^{2m+1}(\theta+1)^2\gamma^2(s)]4t^2+2^{2m}(\theta+1)^2\gamma^2(s)\right|}}$$

$$\left|\frac{a_2}{s}\right|\le\left(\frac{2t}{2^m(1+\theta)\gamma(s)}\right)^2+\frac{2t}{3^m(1+\theta)(\theta+2)\gamma(s)}$$

Remark 7: Let $\in M_\Sigma^m(m,1,\theta,\gamma(s),\phi(z,t))$

$$\left|\frac{a_1}{s}\right|\le\frac{2t\sqrt{2t}}{\sqrt{\left|\left[3^{m+1}(1+\theta)(\theta+2)\gamma(s)-2^{2m+3}(\theta+1)^2\gamma^2(s)\right]4t^2+2^{2m+2}(1+\theta)^2\gamma^2(s)\right|}}$$

$$\left|\frac{a_2}{s}\right|\le\left(\frac{t}{2^m(1+\theta)\gamma(s)}\right)^2+\frac{2t}{3^{m+1}(1+\theta)(\theta+2)\gamma(s)}$$

Corollary 8: Let $f\in M_\Sigma^m(\lambda,\theta,m,\gamma(s))$ then

$$\left|\frac{a_1}{s}\right|\le\frac{2t\sqrt{2t}}{\sqrt{\left[(1+2\lambda)(\theta+1)(\theta+2)3^m-(\lambda^2+5\lambda+2)2^{2m}(1+\theta)^2\right]4t^2+2^{2m}(1+\lambda)^2(1+\theta)^2}}$$

$$\left|\frac{a_2}{s}\right|=\left(\frac{2t}{2^m(1+\lambda)(1+\theta)}\right)+\frac{2t}{3^m(1+2\lambda)(1+\theta)(\theta+2)}$$

Corollary 9: Let $f\in M_\Sigma^m(0,\theta,S,\gamma(0),\phi(z,t))$ then

$$\left|\frac{a_1}{s}\right|\le\frac{2t\sqrt{2t}}{\sqrt{\left|\left[(1+\theta)(\theta+2)3^m-2^{2m+1}(\theta+1)^2\right]4t^2+2^{2m}(1+\theta)^2\right|}}$$

$$\left|\frac{a_2}{s}\right|\le\left(\frac{2t}{2^m(1+\theta)}\right)^2+\frac{2t}{3^m(1+\theta)(\theta+2)}$$

Corollary 10: Let $f\in M_\Sigma^m(0,0,S,\gamma(0),\phi(z,t))$ then

$$\left|\frac{a_1}{s}\right|\le\frac{2t\sqrt{2t}}{\sqrt{\left|\left[2.3^m-2^{2m+1}\right]4t^2+2^{2m}\right|}}$$

$$\left|\frac{a_2}{s}\right|\le\frac{4t^2}{2^{2m}}+\frac{t}{3^m}$$

$$\left|\frac{a_2}{s}\right|\le\frac{4t^2}{4^m}+\frac{t}{3^m}$$

Corollary 11: Let $f\in M_\Sigma^m(1,0,S,\gamma(0),\phi(z,t))$ then

$$\left|\frac{a_1}{s}\right|\le\frac{t\sqrt{2t}}{\sqrt{\left|\left[3^{m+1}.2-2^{m+3}\right]t^2+2^{2m}\right|}}$$

$$\left|\frac{a_2}{s}\right|\le\frac{t^2}{4^m}+\frac{t}{3^{m+1}}$$

Corollary 12: If $f\in M_\Sigma^0(0,0,S,\gamma(0),\phi(z,t))$ we get

$$\left|\frac{a_1}{s}\right|\le 2t\sqrt{2t}$$

$$\left|\frac{a_2}{s}\right|\le 4t^2+t$$

Corollary 13: If $f\in M_\Sigma^0(1,0,S,\gamma(0),\phi(z,t))$ we get

$$\left|\frac{a_1}{s}\right|\le\frac{t\sqrt{2t}}{\sqrt{\left|4-8t^2\right|}}$$

$$\left|\frac{a_2}{s}\right|\le t^2+\frac{t}{3}$$

$$\left|\frac{a_1}{s}\right|\le\frac{2t\sqrt{2t}}{\sqrt{\left|\left[(1+2\beta)(\theta+1)(\theta+2)3_\gamma^m-2^{2m+1}(1+\beta)^2(1+\theta)^2\right]2t^2+2^{2m}(1+\beta)^2(1+\theta)^2\gamma^2(s)\right|}}$$

$$\left|\frac{a_2}{s}\right|\le\left(\frac{2t}{(1+\beta)(1+\theta)\gamma(s)}\right)^2+\frac{4t}{(1+2\beta)(1+\theta)(1+\theta)\gamma(s)}$$

Proof: The prove follows the Theorem 1.

Fekete-Szegö inequality for the Function classes $M_{\Sigma}^{m}(\lambda,\theta,\gamma(s),\phi(z,t))$ then

Theorem 3: Let $f \in M_{\Sigma}^{m}(\lambda,\theta,\gamma(s),S,\phi(z,t))$ and $\mu \in R$. Then one has

$$\left|\frac{a_2}{s}-\mu\frac{a_1}{s}\right| \le \frac{2t}{3^m(1+2\lambda)(1+\theta)(2+\theta)\gamma(s)}, |\mu-1| \le M$$

$$\frac{8|1-\mu|t^3}{\left|\left[3^m(1+2\lambda)(1+\theta)(2+\theta)\gamma(s)-2^{2m}(\lambda^2+5\lambda+2)(1+\theta)^2\gamma^2(s)\right]4t^2+2^{2m}(1+\lambda)^2(1+\theta)^2\gamma^2(s)\right|}, |\mu-1| \ge M$$

where

$$M=\frac{\left|\left[2^{2m}(1+\lambda)^2(1+\theta)^2\gamma^2(s)\right]4t^2+3^m(1+2\lambda)(1+\theta)\gamma(s)-2^{2m}(\lambda^2+5\lambda+2)(1+\theta)^2\gamma^2(s)\right|}{3^m(1+2\lambda)(1+\theta)(2+\theta)\gamma(s)}$$

Proof: From (29) and (36)

$$=U_1(t)\left[\left(h(\mu)+\frac{1}{2(1+2\lambda)(1+\theta)(2+\theta)3^m\gamma(s)}\right)c_2 + \left(h(\mu)-\frac{1}{2(1+2\lambda)(1+\theta)(2+\theta)3^m\gamma(s)}\right)d_2\right]$$

where

$$h(\mu)=\frac{(1-\mu)U_1^2(t)}{2\left[\left((1+2\lambda)3^m(1+\theta)(2+\theta)\gamma(s)-2^{2m}(1+3\lambda)\gamma^2(s)(1+\theta)^2\right)U_1^2(t)-2^{2m}(1+\lambda)^2(1+\theta)^2U_2(t)\right]}$$

Then, by Lemma 1 we conclude that

$$\left|\frac{a_2}{s}-\mu\frac{a_1^2}{s^2}\right| \le \begin{cases} \frac{(1-\mu)U_1^2(t)}{2^{2m}(1+2\lambda)(1+\theta)\gamma(s)}, 0\le|h(\mu)|<\frac{1}{2(1+2\lambda)3^m(1+\theta)(2+\theta)\gamma(s)} \\ 4t|h(\mu)|, |h(\mu)|\ge\frac{1}{2\cdot 3^m(1+2\lambda)(1+\theta)(2+\theta)\gamma(s)} \end{cases}$$

Taking μ = 1 , we have the following corollary

Corollary 14: Let $f \in M_{\Sigma}^{m}(\lambda,\theta,\gamma(s),\phi(z,t))$ then

$$\left|\frac{a_2}{s}-\frac{a_1^2}{s}\right| \le \frac{2t}{3^m(1+2\lambda)(1+\theta)(2+\theta)\gamma(s)}$$

Corollary 15: Let $f \in M_{\Sigma}^{m}(0,\theta,\gamma(s),\phi(z,t))$ then

$$\left|\frac{a_2}{s}-\frac{a_1^2}{s^2}\right| \le \frac{2t}{3^m(1+\theta)(2+\theta)\gamma(s)} \qquad |\mu-1| \le M_1$$

$$\frac{8|1-\mu|t^3}{\left|\left[3^m(1+\theta)(2+\theta)\gamma(s)-2^{2m+1}(1+\theta)^2\gamma^2(s)\right]4t^2+2^{2m}(1+\theta)^2\gamma^2(s)\right|}, \qquad |\mu-1| \ge M_1$$

Where

$$M_1=\frac{\left|\left[2^{2m}(1+\theta)^2\gamma^2(s)\right]4t^2+3^m(1+\theta)(2+\theta)\gamma(s)-2^{2m+1}(1+\theta)^2\gamma^2(s)\right|}{3^m(1+\theta)(2+\theta)\gamma(s)}$$

Theorem 4: Let *f* given by (1) be in the class $F_{\Sigma}^{m}(\beta,\gamma(s),\theta)$ and $\phi \in R$ then

$$\left|\frac{a_2}{s}-\phi\frac{a_1^2}{s^2}\right| \le \frac{4t}{3^m(1+2\beta)(1+\theta)(2+\theta)\gamma(s)}, |\phi-1| \le$$

$$\frac{\left|\left[(1+\beta)^2(1+\theta)^2 2^{2m}(s)\right]4t^2+(1+2\beta)2^{-1}\cdot 3^m(1+\theta)(2+\theta)\gamma(s)-(1+\beta)^2(1+\theta)^2 2^{2m}\gamma^2(s)\right|}{2^{-1}\cdot 3^m(1+2\beta)(1+\theta)(2+\theta)\gamma(s)}$$

$$\frac{8|1-\phi|t^3}{\left|\left[2^{-1}\cdot 3^m(1+2\beta)(1+\theta)(2+\theta)\gamma(s)-(1+\beta)^2(1+\theta)^2 2^{2m}\gamma^2(s)\right]4t^2+(1+\beta)^2(1+\theta)^2 2^{2m}\gamma^2(s)\right|}, |t-1| \ge$$

Corollary 19: If $f \in F_{\Sigma}^{m}(\beta,\gamma(s),\theta)$ and $1 \in R$ then

$$\left|\frac{a_2}{s}-\frac{a_1^2}{s^2}\right| \le \frac{4t}{3^m(1+2\beta)(1+\theta)(2+\theta)\gamma(s)}$$

Corollary 20: If $f \in F_{\Sigma}^{m}(0,\gamma(s),\theta)$ and $\phi \in R$ then

$$\left|\frac{a_2}{s}-\phi\frac{a_1^2}{s^2}\right| \le \frac{4t}{3^m(1+\theta)(2+\theta)\gamma(s)},$$

$$|\phi-1| \le \frac{\left|\left[(1+\theta)^2\gamma^2(s)\right]4t^2+2^{-1}\cdot 3^m(1+\theta)(2+\theta)\gamma(s)-(1+\theta)^2 2^{2m}\gamma^2(s)\right|}{2^{-1}\cdot 3^m(1+\theta)(2+\theta)\gamma(s)}$$

$$\frac{8|1-\mu|t^3}{\left|\left[2^{-1}\cdot 3^m(1+\theta)(2+\theta)\gamma(s)-(1+\theta)^2 2^{2m}\gamma^2(s)\right]4t^2+(1+\theta)^2 2^{2m}\gamma^2(s)\right|}, |\phi-1| \ge$$

Corollary 21: If $f \in F^{\circ}(0,\gamma(s),0)$ and $\phi \in R$ then

$$\left|\frac{a_2}{s}-\phi\frac{a_1^2}{s^2}\right| \le \frac{2t}{\gamma(s)}, \qquad |\phi-1| \le \frac{\left|\left[\gamma^2(s)\right]4t^2+\gamma(s)-\gamma^2(s)\right|}{\gamma(s)}$$

$$\frac{8|1-\mu|t^3}{|[\gamma(s)-\gamma^2(s)]4t^2+\gamma^2(s)|}, |\varphi-1| \ge \frac{|[\gamma^2(s)]4t^2+\gamma(s)-\gamma^2(s)|}{\gamma(s)}.$$

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