



COEFFICIENT BOUNDS FOR THE FUNCTION IN THE CLASS OF MODIFIED HYPERBOLIC TANGENT FUNCTION

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Abstract

In this work, using the principle of subordination in univalent functions, coefficient bounds for the function in the class $S(\alpha, \vartheta)$ of Modified Hyperbolic Tangent Functions were obtained. This work was concluded by determining some functional.

Keywords: Analytic Functions, Hyperbolic Tangent Function, Subordination, Univalent Functions

Introduction

Let E be a unit circle forming the boundary of the region U in the z -plane. The region U is called the unit disk.

$$U = \{z : |z| < 1\}.$$

Let A denote the class of analytic functions on the open unit disk $U = \{z : |z| < 1\}$ normalized by the conditions: $f(0) = 0$, and $f'(0) = 1$, and let S denote the class of all functions in A which are univalent in U . That is,

$$S = \{f: U \rightarrow C: f \text{ is conformal on } U, f(0) = 0, f'(0) = 1\}.$$

We see that every $f \in S$ has a Taylor expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n = z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \dots, |z| < 1, \quad (1)$$

where $a_n \in C, n = 2, 3, 4, \dots$ and $a_1 = 1$, so as to simplify certain formulae for $f \in S$.

Let f and g be analytic functions in U . We say that the function f is subordinate to the function g written as $f < g$, if there exist a

function w , which is analytic in $U, w(0) = 0, |w(z)| < 1, z \in U$, such that

$$f(z) = g(w(z)), \forall z \in U. \quad 2$$

If g is a univalent function in U , then $f < g$, if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

[See Pommerenke (1975)]

The hyperbolic tangent function can be represented in series as seen below:

$$\tanh(z) = z - \frac{z^3}{3} + \frac{2z^5}{15} - \dots = \sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k}-1)B_{2k}z^{2k-1}}{2k!},$$

$\tanh(z)$ is differentiable. For detail information on hyperbolic tangent function.

[See Wolfram]

The set P is the set of all functions of the form $P(z) = 1 + p_1 z + p_2 z^2 + \dots + p_n z^n + \dots = 1 + \sum_{n=1}^{\infty} p_n z^n$ that are analytic in U , and such that for

$$z \in U, \operatorname{Re}(P(z)) > 0$$

See Damer (1983), Godman (1983)].

Following the work of Fadipe-Joseph (2013), we let $h(z)$ be the hyperbolic tangent function and

$$\begin{aligned} \vartheta(z) &= 1 + h(z) \\ &= 1 + \tanh(z) \\ &= 1 + z - \frac{z^3}{3} + \frac{2z^5}{15} - \dots \end{aligned} \tag{3}$$

$\vartheta(z)$ is the modified hyperbolic tangent function.

A function $f \in S$ is said to be in the class $S(\alpha, \vartheta)$, $0 \leq \alpha \leq 1$, if the following subordination holds:

$$(1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)} \right) < \vartheta(z) \tag{4}$$

Motivated by the recent works of Dziok et. al (2015) and Altinkaya et. al (2016) we obtain coefficient bounds for a class of univalent functions using hyperbolic tangent function in the next section as our main results.

Results

Theorem 2.1: If $f(z)$ belongs to the class $S(\alpha, \vartheta)$, then

$$|a_2| \leq \frac{1}{1+\alpha},$$

$$|a_3| \leq \frac{\alpha^2 + 5\alpha + 2}{2(2\alpha^3 + 5\alpha^2 + 4\alpha + 1)},$$

$$|a_4| \leq \frac{8\alpha^4 + 73\alpha^3 + 186\alpha^2 + 101\alpha + 16}{18(6\alpha^5 + 23\alpha^4 + 34\alpha^3 + 24\alpha^2 + 8\alpha + 1)},$$

And

$$|a_5| \leq \frac{664\alpha^6 + 4259\alpha^5 + 9503\alpha^4 + 8153\alpha^3 + 3251\alpha^2 + 620\alpha + 46}{72(48\alpha^8 + 268\alpha^7 + 636\alpha^6 + 835\alpha^5 + 661\alpha^4 + 322\alpha^3 + 94\alpha^2 + 15\alpha + 1)}$$

Proof:

Let $f \in S(\alpha, \vartheta)$. From (2) and (4) we have

$$(1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)} \right) = 1 + w(z) - \frac{(w(z))^3}{3} + \frac{2(w(z))^5}{15} - \dots \tag{5}$$

for some analytic function $w(z)$ such that $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$. It is well-known that if

$$|w(z)| = |c_1z + c_2z^2 + c_3z^3 + \dots| < 1, z \in U, \text{ then } |c_j| \leq 1, \forall j \in \mathbb{N} \tag{6}$$

and

$$|c_2 - \mu c_1^2| \leq \max\{1, |\mu|\}, \forall \mu \in \mathbb{R}. \tag{7}$$

Therefore

$$\begin{aligned} (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)} \right) &= 1 + c_1z^1 + c_2z^2 + \\ &\left(c_3 - \frac{1}{3}c_1^3 \right)z^3 + (c_4 - c_1^2c_2)z^4 + \left(c_5 - c_1^2c_3 - c_1c_2^2 + \right. \\ &\left. \frac{2}{15}c_1^5 \right)z^5 + \dots \end{aligned} \tag{8}$$

Simplifying Equation (8), and equating coefficients, we have

$$(1 + \alpha)a_2 = c_1,$$

$$(2 + 4\alpha)a_3 = c_2 + 3a_2c_1 - 2a_2^2,$$

$$(3 + 9\alpha)a_4 = -(7 + \alpha)a_2a_3 + c_3 + 3a_2c_2 + 4a_3c_1 + 2a_2^2c_1 - \frac{1}{3}c_1^3,$$

$$\begin{aligned} (16\alpha + 4)a_5 &= -a_2c_1^3 + 5a_2a_3c_1 + 2a_2^2c_2 \\ &- c_1^2c_2 + 3a_3^2 + 6a_2a_4 + 5a_4c_1 + 4a_3c_2 + \\ &3a_2c_3 + c_4 - 4a_2a_4\alpha \end{aligned}$$

Now substituting 1 for c_j , for all $j \in \mathbb{N}$ we have the result.

This ends the proof.

Corollary 2.2:

If $f(z)$ belong to the class $S(0, \vartheta)$, then

$$|a_2| \leq 1,$$

$$|a_3| \leq 1,$$

$$|a_4| \leq 8/9$$

$$|a_5| \leq 23/36$$

Corollary 2.3:

If $f(z)$ belong to the class $S(1, \vartheta)$, then

$$|a_2| \leq 1/2,$$

$$|a_3| \leq 1/3,$$

$$|a_4| \leq 2/9$$

$$|a_5| \leq 23/100$$

Fekete-Szegö Inequality for the Function Class $S(\alpha, \vartheta)$.

The Fekete-Szegö functional $|a_3 - \mu a_2^2|$ for normalized univalent functions (1) is well known for its rich history in the theory of geometric functions. It has its origin in the disproof by Fekete and Szegö conjecture of Littlewood and Paley that the coefficients of odd univalent functions are bounded by unity [See Fekete and Szegö (1933)].

Theorem 3.1: Fekete-Szegö inequality for the class $S(\alpha, \vartheta)$.

$$\begin{aligned}
 a_2 &= \frac{c_1}{1+\alpha} \Rightarrow a_2^2 = \frac{c_1^2}{(1+\alpha)^2} \\
 a_3 &= \frac{c_2}{2(1+2\alpha)} + \frac{c_1^2(1+3\alpha)}{2(1+\alpha)^2(1+2\alpha)} \\
 a_3 - \mu a_2^2 &= \frac{c_2}{2(1+2\alpha)} + \frac{c_1^2(1+3\alpha)}{2(1+\alpha)^2(1+2\alpha)} - \frac{\mu c_1^2}{(1+\alpha)^2} \\
 &= \frac{1}{2(1+2\alpha)} \left[c_2 + \frac{c_1^2(1+3\alpha)}{(1+\alpha)^2} - \frac{2\mu c_1^2(1+2\alpha)}{(1+\alpha)^2} \right] \\
 &= \frac{1}{2(1+2\alpha)} \left[c_2 + c_1^2 \left(\frac{(1+3\alpha)}{(1+\alpha)^2} - \frac{2\mu(1+2\alpha)}{(1+\alpha)^2} \right) \right] \\
 |a_3 - \mu a_2^2| &= \left| \frac{1}{2(1+2\alpha)} \left[c_2 + c_1^2 \left(\frac{(1+3\alpha)}{(1+\alpha)^2} - \frac{2\mu(1+2\alpha)}{(1+\alpha)^2} \right) \right] \right| \\
 &= \left| \frac{1}{2(1+2\alpha)} \left[c_2 - c_1^2 \left(\frac{2\mu(1+2\alpha)}{(1+\alpha)^2} - \frac{1+3\alpha}{(1+\alpha)^2} \right) \right] \right| \\
 &\leq \frac{1}{2(1+2\alpha)} \max \left\{ 1, \left| \frac{2\mu(1+2\alpha)}{(1+\alpha)^2} - \frac{1+3\alpha}{(1+\alpha)^2} \right| \right\}.
 \end{aligned}$$

Then in view of (7), it implies that

$$\begin{aligned}
 |a_3 - \mu a_2^2| &\leq \frac{1}{2(1+2\alpha)} \left(\frac{1+3\alpha}{(1+\alpha)^2} \right), \quad \mu = 0 \\
 |a_3 - \mu a_2^2| &\leq \frac{1}{2(1+2\alpha)} \left(\frac{2(1+2\alpha)}{(1+\alpha)^2} \right), \quad \mu = 1 \\
 |a_3 - \mu a_2^2| &\leq \frac{1}{2(1+2\alpha)} \left(\frac{2\mu(1+2\alpha)}{(1+\alpha)^2} - \frac{1+3\alpha}{(1+\alpha)^2} \right), \quad 0 < \mu < 1.
 \end{aligned}$$

Similarly, for $|a_2 a_4 - a_3^2|$

$$\begin{aligned}
 a_2 a_4 &= \frac{c_1}{1+\alpha} \left(\frac{3c_2 - c_1^2}{9(1+3\alpha)} + \frac{2c_1^2}{3(1+\alpha)^2(1+3\alpha)} \right) \\
 &\quad - \frac{\alpha(1-\alpha)}{2(1+\alpha)^2(1+2\alpha)} + \frac{c_1 c_2(1+5\alpha)}{2(1+\alpha)(1+2\alpha)(1+3\alpha)} \\
 &= \frac{3c_1 c_2 - c_1^4}{9(1+\alpha)(1+3\alpha)} + \frac{2c_1^4}{3(1+\alpha)^3(1+3\alpha)} \\
 &\quad - \frac{c_1^4(1-\alpha)}{2(1+\alpha)^4(1+2\alpha)} + \frac{c_1^2 c_2(1+5\alpha)}{2(1+\alpha)^2(1+2\alpha)(1+3\alpha)} \\
 a_3^2 &= \left(\frac{c_2}{2(1+2\alpha)} + \frac{c_1^2(1+3\alpha)}{2(1+\alpha)^2(1+2\alpha)} \right)^2 \\
 &= \frac{c_2^2}{4(1+2\alpha)^2} + \frac{c_1^2 c_2(1+3\alpha)}{4(1+\alpha)^2(1+2\alpha)^2} + \frac{c_1^4(1+3\alpha)^2}{4(1+\alpha)^4(1+2\alpha)^2} \\
 a_2 a_4 - a_3^2 &= \frac{3c_1 c_2 - c_1^4}{9(1+\alpha)(1+3\alpha)} + \frac{2c_1^4}{3(1+\alpha)^3(1+3\alpha)} - \frac{c_1^4(1-\alpha)}{2(1+\alpha)^4(1+2\alpha)} \\
 &\quad + \frac{c_1^2 c_2(1+5\alpha)}{2(1+\alpha)^2(1+2\alpha)(1+3\alpha)} - \frac{c_2^2}{4(1+2\alpha)^2} - \frac{c_1^2 c_2(1+3\alpha)}{4(1+\alpha)^2(1+2\alpha)^2} \\
 &\quad - \frac{c_1^4(1+3\alpha)^2}{4(1+\alpha)^4(1+2\alpha)^2}.
 \end{aligned}$$

Now by making use of (7), we have that

$$\begin{aligned}
 |a_2 a_4 - a_3^2| &\leq \left| \frac{2}{9(1+\alpha)(1+3\alpha)} \left(1 + \frac{3}{(1+\alpha)^2} \right) - \frac{(1-\alpha)}{2(1+\alpha)^4(1+2\alpha)} \right. \\
 &\quad \left. - \frac{(1+5\alpha)}{2(1+\alpha)^2(1+2\alpha)(1+3\alpha)} - \frac{1}{4(1+2\alpha)^2} - \right. \\
 &\quad \left. \frac{(1+3\alpha)^2}{4(1+\alpha)^2(1+2\alpha)^2} \left(\frac{1}{1+3\alpha} + \frac{1}{(1+\alpha)^2} \right) \right|
 \end{aligned}$$

This ends the proof.

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