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CONVERGENCE OF DIFFERENTIAL TRANSFORM METHOD ON NON-LINEAR SYSTEMS OF EQUATIONS

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Abstract

In this paper, the convergence of differential transform method on non-linear systems of ordinary differential equation is considered. The approximate solutions of the initial value problems are calculated in series form with easily computable terms and the convergence of the method is showcased with some examples which are presented to show the efficiency of the method. Analytical solutions of the non-linear equations have been investigated and the results illustrate the reliability and the performance of the differential transform method. The results obtained are in good agreement with the Runge-Kutta method.

Keywords: Differential Transform Method, Non-linear Differential Equations, Runge-Kutta, Analytical Solution.

Introduction

Many problems in mathematical physics, theoretical physics, chemical physics and theoretical biology are modeled by the so-called non-linear partial differential equations. Non-linear equations also cover the cases of the following types: surface waves in compressible fluids, hydro magnetic waves in cold plasma, and acoustic waves in inharmonic crystal etc. However, these equations are difficult to solve analytically and sometimes it is impossible, then application must be made with relevant numerical methods.

A variety of methods, exact, approximate and purely numerical are available for the solution of systems of differential equations. Most of these methods are computationally intensive because they are trial- and –error in nature, or need complicated symbolic computation. In Burrden and Faires, 1993), the Euler method, the Taylor method, Runge-Kutta methods serve as an introduction to numerical method for solving systems of differential equations Shali et al (2016).

The Differential transform method is a numerical as well as analytical method for solving linear, non-linear, shift differential equations, integral equation and ordinary partial differential equations with fast convergence rate and small calculation error. The Differential Transform Method was proposed and first used by Zhou in 1986 for the solution of initial value problem in electric circuit analysis. The method can be used to evaluate the approximate solution by the finite Taylor series and by an iteration procedure described by the transformed equations obtained from the original equation using the operations of differential transformation. The technique is better than other numerical methods since it is free from rounding off error and does not require large computer power. The main advantage of the method is that it can be applied directly to non-linear systems of equations without requiring linearization, discretization or perturbation. It can also be used to obtain both numerical and analytical solution of both linear and non-linear differential equations and is capable of greatly reducing the size of computational work while still accurately providing the series solution with fast convergence rate.

Furthermore, since the use of the technique by Zhou (1986), there are tremendous interests on the application of DTM to solve various scientific problems. Notable among the researchers are Guyer (2001), (Arikoglu and Ozkol, 2005), (Jang and Chen, 1997), (Erturk and Momani, 2007), (Islam et al, 2009), (Abbasov et al, 2008), (Patil and Khambayat, 2014).

This paper presents the differential transform method as a technique for solving nonlinear differential equations without linearization, discretization or perturbation. However, the method can be used to evaluate the approximate solution by the finite Taylor series and by an iteration procedure described by the transformed equation obtained from the original equation using the operations of differential transformation.

The Differential Transform Method

The nth derivative of a function in one-variable of the differential transform is defined as

$$Y(x)=\frac{1}{n!}\left[\frac{d^n y(x)}{dx^n}\right]_{x=x_0} \qquad (1)$$

In equation (1), $Y(x)$ is the original function, $Y(n)$ is the transformed function and the differential inverse transform of $Y(n)$ is defined as

$$y(x)=\sum_{n=0}^{\infty} Y(n)(x-x_0)^n \qquad (2)$$

From equation (1) we obtain

$$y(x)=\sum_{n=0}^{\infty}\frac{(x-x_0)^n}{n!}\left[\frac{d^n y(x)}{dx^n}\right]_{x=x_0} \qquad (3)$$

which implies that the concept of differential transform is derived from Taylor series expansion, but the method does not evaluate the derivative symbolically. Relative derivatives can be calculated by iterative way which is described by the transformed equations of the original function.

1. If $y(t) = u(t) \pm v(t)$ then
   $$Y(k) = U(k) \pm V(k) \qquad (4)$$
2. If $y(t) = au(t)$ then
   $$Y(k) = aU(k) \qquad (5)$$
3. If $y(t)=\dfrac{du(t)}{dt}$ then $\qquad (6)$
   $$Y(K)=(k+1)U(k+1)$$
4. If $y(t)=\dfrac{d^2u(t)}{dt^2}$ then $\qquad (7)$
   $$Y(k)=(k+1)(k+2)U(k+1)$$
5. If $y(t)=\dfrac{d^m u(t)}{dt^2}$ then $\qquad (8)$
   $$Y(k)=(k+1)(k+2)\ldots(k+m)U(k+m)$$
6. If $y(t)=u(t)v(t)$ then $\qquad (9)$
   $$Y(k)=\sum_{\ell=0}^{k}V(\ell)U(k-\ell)$$
7. If $y(t)=t^m$ then $\qquad (10)$
   $$Y(k)=\delta(k-m),\ \delta(k-m)=\begin{cases}1, & if\ k=m\\ 0, & if\ k\neq m\end{cases}$$
8. If $y(t)=\exp(\lambda t)$ then $\qquad (11)$
   $$Y(k)=\frac{\lambda^k}{k!}$$
9. If $y(t)=(1+t)^m$ then
   $$Y(k)=\frac{m(m-1)\ldots(m-k+1)}{k!} \qquad (12)$$
10. If $y(t)=\sin(\omega t+\alpha)$ then $\qquad (13)$
    $$Y(k)=\frac{\omega^k}{k!}\sin\left(\frac{\pi k}{2}+\alpha\right)$$
11. If $y(t)=\cos(\omega t+\alpha)$ then $\qquad (14)$
    $$Y(k)=\frac{\omega^k}{k!}\cos\left(\frac{\pi k}{2}+\alpha\right)$$

12. If $U(k) = D[v(t)]$ and C_1, C_2 are independent of t and k then

$$D[C_1u(t)+C_2v(t)] = C_1U(k)+C_2V(k) \qquad (15)$$

13. If $y(t)=u(t)v(t)$, $u(t)=D^{-1}[U(k)]$, $v(t)=D^{-1}[V(k)]$,

$$D[z(t)]=D[u(t)v(t)]=U(k)\otimes V(k)=\sum_{\ell=0}^{k}V(\ell)U(k-\ell) \qquad (16)$$

where $\otimes$ denotes the convolution.

Therefore, the transform of $U^m(t)$, where m is a positive integer, can be obtained as follows;

$$D[U^m(t)]=U^m(k)=U^{m-1}(k)\otimes U(k)=\sum_{\ell=0}^{k}U^{m-1}(\ell)U(k-\ell) \qquad (17)$$

Numerical Examples

In this section, the differential transformation technique is applied to solve different systems (linear, nonlinear and shift) of differential equations

Example 1:

Consider the non-linear system

$$\begin{aligned} y_1'(x) &= -y_1(x) \\ y_2'(x) &= y_1(x) - y_2^2(x) \\ y_3'(x) &= y_2^2(x) \end{aligned} \qquad (18)$$

with initial condition

$y_1(0) = 1,\ y_2(0) = 0$ *and* $y_3(0) = 0$ (19)

Solution

Taking the differential transformation of equations (18) in conjunction with equation (8), we have

$$Y_1(k+1) = -\frac{1}{(k+1)}[Y_1(k)] \qquad (20)$$

$$Y_2(k+1) = \frac{1}{(k+1)}\left[Y_1(k) - \sum_{\ell=0}^{k} Y_1(\ell)Y_2(k-\ell)\right] \qquad (21)$$

$$Y_3(k+1) = \frac{1}{(k+1)}\sum_{\ell=0}^{k} Y_2(\ell)Y_2(k-\ell) \qquad (22)$$

$$Y_1(0) = 1 \qquad (23)$$

$$Y_2(0) = 0 \qquad (24)$$

$$Y_3(0) = 0 \qquad (25)$$

The set of values $Y_1(k+1),\ Y_2(k+2)$ *and* $Y_3(k+3)$ *of* $y_1(x),\ y_2(x)$ with the differential transformation method are present in Table 1.

For k = 0, 1, 2. . . n, ... , the solution of the given non – linear differential systems (20), (21) and (22) can be written in the closed form as follows:

Table 1: Differential transformation values of example 1 for k = 0, 1, 2, . . .

k	$y_1(k+1)$	$y_2(k+1)$	$y_3(k+1)$
0	1	0	0
1	1/2	-1/2	0
2	-1/6	-1/6	1/3
3	1/24	5/24	-1/4
4	-1/120	1/40	-1/6
5	1/720	-71/720	7/72
6	-1/5040	19/1008	-47/2520

The solution is given as follows:

$$\begin{aligned} y_1(x) &= 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{120}x^5 + \frac{1}{720}x^6 - \frac{1}{5040}x^7 + \frac{1}{40320}x^8 - \frac{1}{362880}x^9 + \ldots \\ y_2(x) &= x - \frac{1}{2}x^2 - x^3 + \frac{5}{24}x^4 + \frac{1}{40}x^5 - \frac{71}{720}x^6 + \frac{19}{1008}x^7 - \frac{1469}{40320}x^8 + \ldots \qquad (26) \\ y_3(x) &= \frac{1}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{60}x^5 + \frac{7}{72}x^6 - \frac{47}{2520}x^7 - \frac{7}{192}x^8 + \ldots \end{aligned}$$

Example 2

Consider the following system of non-homogeneous differential equation

$$\begin{aligned} y_1'(x) &= y_3(x) - \cos(x) \qquad (27) \\ y_2'(x) &= y_3(x) - e^x \\ y_3'(x) &= y_1(x) - y_2(x) \end{aligned}$$

with initial condition

$$y_1(0) = 0, \quad y_2(0) = 0 \ \text{ and } \ y_3(0) = 2 \qquad (28)$$

The exact solution of this problem is

$$y(x) = (y_1(x),\ y_2(x),\ y_3(x)) = (e^x, \sin(x),\ e^x + \cos(x)) \qquad (29)$$

Choosing x_0 = 0 equations (27) are transformed as follows;

$$(k+1)Y_1(k+1) - Y_3(k) = -\frac{1}{k!}\cos\left(\tfrac{k\pi}{2}\right) \quad (30)$$

$$(k+1)Y_2(k+1) - y_3(k) = -\frac{1}{k} \quad (31)$$

(32)

$$(k+1)Y_3(k+1) - Y_1(k) + Y_2(k) = 0 \quad (33)$$

Table 2

Differential transformation values of example 2 for k = 0, 1, 2, . . .

k	$y_1(k+1)$	$y_2(k+1)$	$y_3(k+1)$
0	0	0	2
1	1	1	1
2	1/2!	0	0
3	1/3!	-1/3!	1/3!
4	1/4!	0	1/4!
5	1/5!	1/5!	0
6	1/6!	0	1/6!

Therefore, the solution of the equation (1) is given by

$$\begin{aligned} y_1(x) &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \ldots = e^x \\ y_2(x) &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \ldots = e^x \\ y_3(x) &= \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots\right) + \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \ldots\right) = e^x + \sin(x) \end{aligned} \quad (34)$$

Example 3

Consider the following non-linear system of differential equations

$$\begin{aligned} y_1'(x) &= 2e^{4x}y_4^2(x) \\ y_2'(x) &= y_1(x) - y_3(x) + \cos(x) - e^{2x} \\ y_3'(x) &= y_2(x) - y_4(x) + e^{-x} - \sin(x) \\ y_4'(x) &= -e^{-5x}y_1^2(x) \end{aligned} \quad (35)$$

with the condition

$y_1(0) = 1$, $y_2(0) = 1$ and $y_3(0) = 0$ $y_4(0) = 2$ (36)

The exact solution of this problem is,

$$y(x) = (y_1(x),\ y_2(x),\ y_3(x), y_4 x)) = (e^x,\ \sin(x) + osx),\ \sin x),\ e^{-x}) \quad (37)$$

Equation (35) and (36) are transformed as follows;

$$(k+1)Y_1(k+1) = 2\sum_{k_2=1}^{k}\sum_{k_1=0}^{k_2}\frac{4^{k_1}}{k_1!}Y_4(k_2 - k_1)Y_4(k - k_2) \quad (38)$$

$$(k+1)Y_2(k+1) = y_3(k) = Y_1(k) - Y_3(k) + \frac{1}{k!}\cos\frac{k\pi}{2} - \frac{2^k}{k!} \quad (39)$$

$$(k+1)Y_3(k+1) = Y_2(k) - Y_4(k) + \frac{(-1)^k}{k!} - \frac{1}{k!}\sin\frac{k\pi}{2} \quad (40)$$

$$(k+1)Y_4(k+1) = 2\sum^{k}\sum^{k_2}\frac{(-5)^{k_1}}{k_1!}Y_1(k_2 - k_1)Y_1(k - k_2) \quad (41)$$

$y_1(0) = 1$, $y_2(0) = 1$ and $y_3(0) = 0$ $y_4(0) = 1$ (42)

Therefore, the solution of equation (35) is given by

$$\begin{aligned} y_1(x) &= 1 + 2x + 2x^2 + \frac{8}{3!}x^3 + \ldots = e^{2x} \\ y_2(x) &= \left[x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \ldots\right] + \left[1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \ldots\right] = \sin(x) + \cos(x) \\ y_3(x) &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \ldots = \sin(x) \\ y_4(x) &= 1 - x - \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \ldots = e^{-x} \end{aligned} \quad (43)$$

Table 3

Differential transformation values of example 2 for k = 0, 1, 2, . . .

k	$y_1(k+1)$	$y_2(k+1)$	$y_3(k+1)$	$y_4(k+1)$
0	1	1	0	1
1	2	1	1	-1
2	2	-1/2!	0	1/2!
3	8/3!	-1/3!	-1/3!	-1/3!
4	16/4!	1/4!	0	1/4!
5	32/5!	-1/5!	1/5!	-1/5!
6	64/6!	-1/6!	0	1/6!

Table 4

Errors involved with the differential transform method along with the result obtained by the Runge-Kutta fourth-order method in Example 1.

x_i	$\lvert y_1(x_i) - RKM \rvert$	$\lvert y_2(x_i) - RKM \rvert$	$\lvert y_3(x_i) - RKM \rvert$
0.00	0.00	0.00	0.00
0.02	0.0126781	0.129e-3	0.852e-3
0.04	0.361434e-3	0.66543334e-3	0.33481333e-1
0.06	0.305466667e-2	0.19796767e-1	0.27659e-1
0.08	0.75222233e-1	0.48702239e-3	0.602158e-4
0.10	0.1112612e-1	0.54001111e-2	0.211622e-3

Table 5

Errors involved with the differential transform method along with the result obtained by the Runge-Kutta fourth-order method in Example 2.

x_i	$\lvert y_1(x_i) - RKM \rvert$	$\lvert y_2(x_i) - RKM \rvert$	$\lvert y_3(x_i) - RKM \rvert$
0.00	0.00	0.00	0.00
0.02	0.101012	0.129e-3	0.00
0.04	0.1000000e-6	0.66543334e-3	0.100000e-5
0.06	0.44544433e-1	0.19796767e-1	0.166445
0.08	0.33400001e-8	0.1000000e-5	0.3300004e-8
0.10	0.88777667e-4	0.922211e-3	0.111345e-3

Table 6

Errors involved with the differential transform method along with the result obtained by the Runge-Kutta fourth-order method in Example 3.

x_i	$\lvert y_1(x_i) - RKM \rvert$	$\lvert y_2(x_i) - RKM \rvert$	$\lvert y_3(x_i) - RKM \rvert$	$\lvert y_4(x_i) - RKM \rvert$
0.00	0.00	0.00	0.00	0.00
0.02	0.100000e-8	0.100000e-7	0.100000e-7	0.011234
0.04	0.111000e-3	0.792002e-3	0.100000e-6	0.44415e-3
0.06	0.1220023	0.5334556e-2	0.5334556e-2	0.50162311e-2
0.08	0.10000e-3	0.3846434e-3	0.12004e-5	0.54644e-3
0.10	0.40355089	0.79877667e-3	0.8211e-3	0.1054767e-3

Conclusion

The differential transformation method is a powerful tool which is used to find analytical solution in case of linear and non-linear systems of differential equations. The method has been successfully applied to non-linear system of differential equations. The present method reduces the computational difficulties of the other traditional methods and all the calculations can be made with simple manipulations. The method is better than numerical methods, since it saves time and space, free from rounding off error, easy to implement and does not require large computer power. Some examples were tested by applying the DTM and the results have shown remarkable performance. Therefore, this method can be applied to many non-linear integral and differential equations without linearization, discretization or perturbation.

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