



ON THE TRANSVERSE VIBRATIONS OF ROTATING TIMOSHENKO BEAMS SUBJECTED TO TWO CONCENTRATED MOVING LOADS

Adedowole, A.

Department of Mathematical Sciences, Adekunle Ajasin University Akungba-Akoko,
PMB 001, Akungba-Akoko Ondo State Nigeria

Corresponding author's email: alimi.adedowole@aaau.edu.ng

Abstract:

This study focused on the response of rotating Timoshenko beams subjected to two concentrated moving loads at distance d apart. The versatile Galerkin's method has been developed and applied to treat the problem of the dynamic deflections of beam structure and the resulting simultaneous equations are reduced to simple algebraic equations via Laplace transform. Numerical analyses in plotted curves are presented. The analyses depict interesting results on the effect of some structural parameters such as foundation moduli and prestressed forces on the dynamic behaviour of rotating Timoshenko beams under the actions of moving loads. The effects of the two loads 1P and 0P, and distance d apart of the two loads were also considered. It was observed that the dynamic response amplitudes of beam decreased as the distance between the loads increases. The resonance condition of the dynamical systems was also established.

Keywords: Resonance, Foundation Stiffness, Prestressed, Transverse Response, Galerkin's Method.

Introduction

The response of structural members such as beams and plates subjected to statics loads or moving loads has been area of interest to researchers across the globe in the field of Engineering and Mathematical Physics. The most interesting aspect is when such structures are subjected to moving loads. Among these notable researchers are Fryba (1972), Stanisic *et al* (1974), Oni and Omolofe (2005), Oni. and Adedowole (2008), and Omolofe and Adedowole (2017).

A simply supported beam subjected to a constant moving force at uniform speed was considered by Krylov (1905) who used the method of expansion of the associated Eigen

modes. He assumed the mass of the loads to be smaller than that of the beam. Djondjorov (1995) studied the problem of vibration of multi-span Timoshenko beam. His study shows that the effects of rotatory inertia and shear deformation cause the modal frequencies of the Timoshenko beam to be less than those of Bernoulli-Euler beam. Meera *et al.* (2007) conducted a free vibration analysis for uniform Timoshenko beams using the coupled displacement field method. Ogunbamike (2012) studied the dynamic response of a uniform deep beam resting on a Winkler elastic foundation and excited by a moving load. Mahmoud *et al.* (2013) applied the differential transformation method (DTM) for free vibration analysis of

beams with uniform and non-uniform cross sections.

Adedowole (2016) worked on flexural motions under moving distributed masses of Beam- type structures on Vlasor foundation and having time dependent boundary conditions. Adedowole and Famuwagun (2017) also consider the dynamic response under travelling loads of simply supported non prismatic beam resting on variable elastic foundation Method of Laplace Integral transforms was employed to solve this initial valued problem to obtain the desired approximate solutions of the reduced equations for the transverse displacement response of the beam dynamical problem. Analyses showed that higher values of the axial force and foundation stiffness decrease the transverse displacement response of the non prismatic beam under the action of travelling loads resting on variable elastic foundation.

Recently, Jimoh (2017) studied the motion of non-uniformly pre-stressed tapered beams with exponentially varying thickness resting on Vlasov foundation under variable harmonic load moving with constant velocity. Very recently, Jimoh *et al.* (2018) studied the dynamic response of non-uniformly pre-stressed thick beam under distributed moving load travelling at varying velocity. Techniques based on the method of Galerkin with the series representation of Heaviside function, was first used to transform the equation and thereafter the transformed equations were solved using Strubles asymptotic method and Laplace transformation techniques in conjunction with convolution theory. The displacement response for moving distributed force and moving distributed mass models for the dynamical problem are calculated for various time t and presented

in plotted curves.

More recently, Adeoye and Awodola (2018) worked on dynamic response to moving distributed masses of pre-stressed uniform rayleigh beam resting on variable elastic pasternak foundation. Adedowole and Jimoh (2018) considered Dynamics analysis of a damped non uniform beam subjected to loads moving with variable velocity. they used a procedure involving the Galerkin's method and integral transform technique to solve the problem of a non-uniform beam when it is subjected to constant and harmonic variable magnitude moving loads. In particular, analytical solution in series form is obtained for the deflection of the elastic beam and the effects of foundation stiffness K and the axial force N on the vibrating system were investigated. Adedowole (2018) investigated on the response of non-prismatic rotating Timoshenko beam under the actions of concentrated loads travelling at time dependent speeds.

However, it is remarked at this juncture that in most of the existing literature in dynamics of structure under moving loads, study on load more than one is very scanty. Shahin (2010) studied the response of a simply-supported beam on elastic foundation to repeated moving concentrated load by means of the Fourier sine transformation. The effects of some important parameters, such as the foundation stiffness, distance parameter and the travelling speed were studied.

The aim of this paper is to obtain the analytical solutions of the governing fourth order partial differential equations with variable and singular coefficients of elastic beams under two travelling loads.

This paper therefore, investigates the transverse motions of Timoshenko beam under the actions of two concentrated moving loads. It was observed that the dynamic

response amplitudes of beam decrease as the distance d between the loads increases.

Problem Formulation

This paper considers the dynamic behaviour of a rotating Timoshenko beam resting on a elastic foundation when it is under the action of two moving loads. The beam is assumed to maintain contact with the subgrade reaction modulus E_f and that there is no frictional force at the interface. The deflection $w(x,t)$ from the equilibrium and the rotation $u(x,t)$ of the beam under the action of moving load is described by the system of partial differential equations

$$\mu_0 w_{tt}(x,t) - K^*GA(w_x(x,t) - u_x(x,t)) + E_f w(x,t) = P(x,t) + N(x)w_x(x,t) \tag{1}$$

$$H(x) + K^*GA(w_x(x,t) - u(x,t)) - I\rho u_{tt}(x,t) = 0 \tag{2}$$

Where K^* is a constant dependent on the shape of the cross-section, G is the modulus of elasticity in the shear, A is the cross-sectional area, $P(x,t)$ are the moving concentrated forces acting on the beam, μ is the mass of the beam per unit length L , W is the vertical response of the beam, I is the moment of inertia of the beam cross-section, d is the distance, N is the pre-stress and E_f is the constant elastic foundation.

The flexural moment acting on the beam cross section is related to the vertical response to rotation as

$$H(x) = \frac{\partial}{\partial x} (D_x(x,t)u_x) \tag{3}$$

$D_x(x,t)$ is the flexural stiffness of the beam given as

$$D_x(x,t) = \gamma I \tag{4}$$

Where γ is the Young Modulus

The Boundary Conditions

The boundary conditions depend on the constraints at the beam ends. For a beam whose length is L , the vertical displacement at the beam ends are given as

$$w(0,t) = u(0,t) = 0, w(L,t) = u(L,t) = 0 \tag{5}$$

It is assumed that the initial conditions are

$$w(x,t) = 0 = w_t(x,0) \text{ and } u(x,0) = 0 = u_t(x,0) \tag{6}$$

Non-uniformity of prestressed

The distribution of the non-characteristics may be assumed as power functions. The parameters α and n , are used to approximate the actual non uniformity of the beam prestressed given as

$$N(x) = N_f(1 + \alpha x)^n \tag{7}$$

N_f is the beam characteristics at $x = 0$

The constant vertical excitation acting on the beam is chosen as

$$P(x,t) = P_1\delta(x - vt) + P_0\delta\{x - (vt + d)\} \tag{8}$$

Where d the distance between the two given loads, P_1 and P_0 are the loads

The concentrated load is assumed to be of mass M and the time t is assumed to be limited to that interval of time within the mass on the beam, that is;

$$0 \leq t \leq \frac{l - d}{v} \tag{9}$$

Substituting equations (7) and (8) into equation (1) and (2) taking $1=n$ for simplicity yield

$$\mu_0 w_{tt}(x,t) - K^*GA(w_x(x,t) - u_x(x,t)) - N_f(1 + \alpha x)w_x(x,t) + E_f w(x,t) = P_1\delta(x - vt) + P_0\delta\{x - (vt + d)\} \tag{13}$$

and

$$\frac{\partial}{\partial x} (I\rho u_x(x,t)) + K^*GA(w_x(x,t) - u(x,t)) - I\rho u_{tt}(x,t) = 0 \tag{14}$$

Now we seek the closed form solution to the simultaneous second order partial differential equations (13) and (14). Consequently, an approximate analytical solution is desirable to obtain some vital information about the vibrating system.

Solution Technique

Methodology: The governing equation of the problem is a fourth order partial differential equation. In order to solve this problem, elegant technique called Galerkin's Method is used to reduce the governing fourth order partial differential equations with variable and singular coefficients to a sequence of

second order ordinary differential equations.

In order to solve the beam problem above, we shall use the versatile solution technique called Galerkin's method often used in solving diverse problems involving mechanical vibrations Omolofe 2017. This technique requires that the solutions equations of the form

$$\Gamma w(x,t) - P(x,t) = 0 \tag{15a}$$

G is the differential operator, w is the structural displacement and P is the traverse load acting on the structure. To this effect, the solutions of the system of equations (13) and (14) are expressed as

$$w(x,t) = \sum_{i=1}^n e_i(t) z_i(x) \tag{15}$$

and

$$u(x,t) = \sum_{i=1}^n y_i(t) r_i(x) \tag{16}$$

where the functions $z_i(x)$ and r_i are chosen to satisfy the pertinent boundary conditions. Thus, substituting equations (15) and (16) into the coupled simultaneous ordinary differential equations (13) and (14) we obtain

$$\begin{aligned} & \sum_{i=1}^n \left\{ \mu \ddot{e}_i(t) z_i(x) - K^* GA(e_i(t) z_i''(x) - y_i(t) r_i'(x)) \right. \\ & \left. - N e_i(t) z_i''(x) + E_f(x) e_i(t) z_i(x) \right\} \\ & = P_1 \delta(x-vt) + P_0 \delta\{x-(vt+d)\} \end{aligned} \tag{17}$$

and

$$\begin{aligned} & \sum_{i=1}^n \left\{ \gamma \left(y_i(t) r_i''(x) + y_i(t) r_i'(x) \right) \right. \\ & \left. + K^* GA(e_i(t) z_i'(x) - y_i(t) r_i(x)) - I \rho \ddot{y}_i(t) r_i(x) \right\} = 0 \end{aligned} \tag{18}$$

To determine $e_i(t)$ and $y_i(t)$, the expressions on the left hand sides of equations (17) and (18) are required to be orthogonal to the functions $e_k(t)$ and $y_k(t)$ respectively. Thus,

$$\begin{aligned} & \int_0^L \int_{-1}^n \left\{ \mu \ddot{e}_i(t) z_i(x) - K^* GA(e_i(t) z_i''(x) - y_i(t) r_i'(x)) \right. \\ & \left. - N_f e_i(t) z_i''(x) + E_f(x) e_i(t) z_i(x) \right\} \\ & - P_1 \delta(x-vt) + P_0 \delta\{x-(vt+d)\} z_k(x) dx = 0 \end{aligned} \tag{20}$$

and

$$\begin{aligned} & \int_0^L \int_{-1}^n \left\{ \gamma \left(y_i(t) r_i''(x) + y_i(t) r_i'(x) \right) \right. \\ & \left. + K^* GA(e_i(t) z_i'(x) - y_i(t) r_i(x)) - I \rho \ddot{y}_i(t) r_i(x) \right\} r_k(x) dx = 0 \end{aligned} \tag{21}$$

Equation (20) and (21) after some rearrangements yield

$$\sum_{i=1}^n [q_a(i,k) \ddot{e}_i(t) + q_b(i,k) e_i(t) + q_c(i,k) y_i(t)] = q_d \tag{22}$$

and

$$\sum_{i=1}^n [a_1(i,k) \ddot{y}_i(t) + a_2(i,k) e_i(t) + a_3(i,k) y_i(t)] = 0 \tag{23}$$

where

$$\begin{aligned} q_a(i,k) &= \mu \int_0^L z_i(x) z_k(x) dx, \\ q_b(i,k) &= \int_0^L (-K^* GA z_i''(x) - N_f (1 + \alpha x) z_i'(x) + E_f z_i(x)) \\ & z_k(x) dx, \quad q_c(i,k) = K^* GA \int_0^L r_i'(x) z_k(x) dx, \\ q_d &= \int_0^L P_1 \delta(x-vt) + P_0 \delta\{x-(vt+d)\} z_k(x) dx \end{aligned} \tag{24a}$$

$$\begin{aligned} a_1(i,k) &= -I \rho \int_0^L r_i(x) r_k(x) dx, \quad a_2(i,k) = K^* GA \int_0^L z_i'(x) r_k(x) dx, \\ a_3(i,k) &= \int_0^L [\gamma (r_i(x) + r_i'(x)) K^* GA r_i(x) + r_k(x) dx \end{aligned} \tag{24b}$$

Since our beam has simple supports at both ends $x = 0$ and $Lx=$, we therefore choose the functions $z_i(x)$ and $h_i(x)$ to be

$$z_i(x) = \sin \frac{i\pi x}{L} \quad \text{and} \quad h_i(x) = \cos \frac{i\pi x}{L} \tag{25}$$

Thus, in view of (25), integrals (24) are evaluated to yield

$$\begin{aligned} q_a(i,k) &= \mu I_1, \\ q_b(i,k) &= \left(\frac{i\pi}{L} \right)^2 (K^* GA I_1 + N_f (I_1 + \alpha I_2)) + E_f I_1, \\ q_c(i,k) &= -K^* GA \left(\frac{i\pi}{L} \right) I_1, \\ q_d &= P_1 \sin \phi + P_0 \sin \frac{m\pi c(t+d)}{L} \end{aligned} \tag{26}$$

$$\begin{aligned} a_1(i,k) &= -I \rho I_2, \quad a_2(i,k) = K^* GA \left(\frac{i\pi}{L} \right) I_2, \\ a_3(i,k) &= - \left(\gamma \left(\frac{i\pi}{L} \right)^2 + K^* GA \right) I_2 \end{aligned} \tag{27}$$

where

$$I_1 = \int_0^l \sin \frac{i\pi x}{L} \sin \frac{k\pi x}{L} dx, \quad I_2 = \int_0^l \cos \frac{i\pi x}{L} \cos \frac{k\pi x}{L} dx, \quad (28)$$

Considering only *i*th concentrated moving load when moving loads are of different magnitudes

$$q_d = H_p \sin \phi t + H_a \cos \phi t \quad (29)$$

Where

$$H_p = P_1 + P_0 b_0, \quad H_a = P_0 a_0 \quad (30)$$

equation (22) and (23) can be simplified further to give

$$q_a(i, k) \ddot{e}_i(t) + q_b(i, k) \dot{e}_i(t) + q_c(i, k) y_i(t) = H_p \sin \phi t + H_a \cos \phi t \quad (31)$$

and

$$a_1(i, k) \ddot{y}_i(t) + a_2(i, k) \dot{y}_i(t) + a_3(i, k) y_i(t) = 0 \quad (32)$$

Where

$$\phi = \frac{m\pi t}{L}, \quad a_0 = \sin \frac{m\pi d}{L} \quad \text{and} \quad b_0 = \cos \frac{m\pi d}{L} \quad (33)$$

In what follows we subject the system of ordinary differential equations (31) and (32) to a Laplace transform defined as

$$(\sim) = \int_0^\infty (\cdot) e^{-st} dt \quad (35)$$

where *s* is the Laplace parameter. In conjunction with the initial conditions define in (6), yields the following algebraic simultaneous equation

$$[q_a(i, k)s^2 + q_b(i, k)]e_i(s) + q_c(i, k)y_i(s) = H_p \frac{\phi}{S^2 + \phi^2} + H_a \frac{S}{S^2 + \phi^2} \quad (36)$$

And

$$[a_1(i, k)s^2 + a_3(i, k)]y_i(s) + a_2(i, k)e_i(s) = 0 \quad (37)$$

In order to solve the above system, the

following representations are made

$$\begin{aligned} \psi_0 &= \begin{vmatrix} q_a(i, k)s^2 + q_b(i, k) & q_c(i, k) \\ a_2(i, k) & a_1(i, k)s^2 + a_3(i, k) \end{vmatrix} \\ \psi_1 &= \begin{vmatrix} \psi_a(1,1) & q_c(i, k) \\ 0 & a_1(i, k)s^2 + a_3(i, k) \end{vmatrix} \\ \psi_2 &= \begin{vmatrix} q_a(i, k)s^2 + q_b(i, k) & \psi_a(1,1) \\ a_2(i, k) & 0 \end{vmatrix} \end{aligned} \quad (38)$$

Where

$$\psi_a(1,1) = H_p \frac{\phi}{S^2 + \phi^2} + H_a \frac{S}{S^2 + \phi^2} \quad (40)$$

Thus

$$e_i(s) = \frac{[a_1(i, k)s^2 + a_3(i, k)]\psi_a(1,1)}{d_1 s^4 + d_2 s^2 + d_3} \quad (41)$$

And

$$y_i(s) = \frac{-a_1(i, k)\psi_a(1,1)}{d_1 s^4 + d_2 s^2 + d_3} \quad (42)$$

Furthermore, equations (41) and (42) can be re-written in the form

$$e_i(s) = \frac{[a_1(i, k)s^2 + a_3(i, k)]\psi_a(1,1)}{d_1 (s^2 + \eta_1^2)(s^2 + \eta_1'^2)} \quad (43)$$

and

$$y_i(s) = \frac{a_2(i, k)\psi_a(1,1)}{d_1 (s^2 + \eta_1^2)(s^2 + \eta_1'^2)} \quad (44)$$

Where

$$\begin{aligned} \eta_1 &= \sqrt{\frac{d_2}{d_1} + \left(\frac{d_2}{4d_1^2} - \frac{d_3}{d_1}\right)^{\frac{1}{2}}}, \quad \eta_2 = \sqrt{\frac{d_2}{d_1} - \left(\frac{d_2}{4d_1^2} - \frac{d_3}{d_1}\right)^{\frac{1}{2}}} \\ d_1 &= q_a(i, k)a_1(i, k), \quad d_2 = q_a(i, k)a_3(i, k) + q_b(i, k)a_1(i, k) \\ \text{and} \quad d_3 &= q_a(i, k)a_3(i, k) - q_c(i, k)a_2(i, k) \end{aligned}$$

Solving equations (43) and (44) further, one obtains

$$e_i(s) = \frac{1}{d_1} \left[\frac{1}{(s^2 + \eta_1^2)} * \frac{[a_1(i,k)\eta_1^2 + a_3(i,k)]}{(\eta_1^2 - \eta_2^2)} - \frac{1}{(s^2 + \eta_2^2)} * \frac{[a_1(i,k)\eta_1^2 + a_3(i,k)]}{(\eta_1^2 - \eta_2^2)} \right] * \left\{ H_p \frac{\phi}{S^2 + \phi^2} + H_a \frac{S}{S^2 + \phi^2} \right\} \tag{46}$$

And

$$y_i(s) = Q_i \left[\frac{1}{(s^2 + \eta_1^2)} - \frac{1}{(s^2 + \eta_2^2)} \right] * \left\{ H_p \frac{\phi}{S^2 + \phi^2} + H_a \frac{S}{S^2 + \phi^2} \right\} \tag{47}$$

Some simplifications and rearrangements of equations (46) and (47) yield

$$e_i(s) = \frac{H_p}{d_1} \left[\frac{a_1(i,k)\eta_1^2 + a_3(i,k)}{(\eta_1^2 - \eta_2^2)} \right] \left[\left[\frac{1}{(s^2 + \eta_1^2)} * \frac{\phi}{S^2 + \phi^2} \right] - \left[\frac{1}{(s^2 + \eta_2^2)} * \frac{\phi}{S^2 + \phi^2} \right] \right] + \frac{H_a}{Q_i} \left[\left[\frac{1}{(s^2 + \eta_1^2)} * \frac{s}{S^2 + \phi^2} \right] - \left[\frac{1}{(s^2 + \eta_2^2)} * \frac{s}{S^2 + \phi^2} \right] \right] \tag{48}$$

$$y_i(s) = H_p Q_i \left[\frac{1}{(s^2 + \eta_1^2)} * \frac{\phi}{S^2 + \phi^2} \right] - \left[\frac{1}{(s^2 + \eta_2^2)} * \frac{\phi}{S^2 + \phi^2} \right] + H_a Q_i \left[\frac{1}{(s^2 + \eta_1^2)} * \frac{s}{S^2 + \phi^2} - \frac{1}{(s^2 + \eta_2^2)} * \frac{s}{S^2 + \phi^2} \right] \tag{49}$$

Where

$$Q_i = \frac{a_2(i,k)}{d_1(\eta_1^2 - \eta_2^2)} \tag{49b}$$

In order to obtain the Laplace inversion of equations (48) and (49), we shall adopt the following representations

$$g_1(s) = \frac{\phi}{S^2 + \phi^2}, \quad g_2(s) = \frac{S}{S^2 + \phi^2},$$

$$f_1(s) = \frac{\eta_1}{s^2 + \eta_1^2} \text{ and } f_2(s) = \frac{\eta_2}{s^2 + \eta_2^2} \tag{50}$$

So that the Laplace inversion of each term of the RHS (48) and (49) is the convolution of f_i and g_j defined by as

$$f_i * g_j = \int_0^t f_i(t-u)g_j(u)du, \quad i=1,2 \text{ and } j=1,2 \tag{51}$$

Thus the Laplace inversion of (48) and (49) are respectively given as

$$e_i(t) = \frac{[a_1(i,k)\eta_1^2 + a_3(i,k)]}{(\eta_1^2 - \eta_2^2)} \frac{H_p}{d_1} (G_1 - G_2) + \frac{H_a}{d_1} (G_3 - G_4) \tag{52}$$

$$y_i(t) = \frac{a_2(i,k)}{(\eta_1^2 - \eta_2^2)} \frac{H_p}{d_1} [G_1 - G_2] + \frac{a_2(i,k)}{(\eta_1^2 - \eta_2^2)} \frac{H_a}{d_1} [G_3 - G_4] \tag{53}$$

Where

$$G_1 = \frac{1}{\eta_1} \int_0^L \sin \eta_1(t-u) \sin \phi u du, \quad G_2 = \frac{1}{\eta_2} \int_0^L \sin \eta_2(t-u) \sin \phi u du$$

$$G_3 = \frac{1}{\eta_1} \int_0^L \sin \eta_1(t-u) \cos \phi u du, \quad G_4 = \frac{1}{\eta_2} \int_0^L \sin \eta_2(t-u) \cos \phi u du \tag{54}$$

In what follows, we shall evaluate integrals (54) above and to do so, we note the following trigonometric identities namely

$$\frac{1}{g} \int_0^L \sin \theta(t-u) \sin \Omega u du = \frac{\theta \sin \theta}{g^2 - \Omega^2} * \left[\sin \Omega \sin \theta - \frac{\Omega}{g} (\cos \Omega \cos \theta - 1) \right]$$

$$+ \frac{\theta \cos \theta}{g^2 - \Omega^2} \left[\sin \Omega \cos \theta - \frac{\Omega}{g} (\cos \Omega \sin \theta) \right]$$

$$\frac{1}{g} \int_0^L \sin \theta(t-u) \cos \Omega u du = \frac{\Omega \sin \theta}{g^2 - \Omega^2} * \left[\cos \theta \sin \Omega - \frac{\theta}{\Omega} (\sin \theta \cos \Omega) \right]$$

$$+ \frac{\Omega \cos \theta}{\Omega^2 - g^2} \left[\sin \theta \sin \Omega - \frac{\theta}{\Omega} (\cos \theta \cos \Omega - 1) \right] \tag{55}$$

In view of (55), integrals (54) are thus evaluated and we have

$$\frac{\eta_1 \cos \eta_1 t}{\eta_1^2 - \phi^2} \left[\sin \phi t \cos \eta_1 t - \frac{\phi}{\eta_1} (\cos \phi t \sin \eta_1 t) \right]$$

$$G_2 = \frac{\eta_2 \sin \eta_2 t}{\eta_2^2 - \phi^2} \left[\sin \phi t \sin \eta_2 t - \frac{\phi}{\eta_2} (\cos \phi t \cos \eta_2 t - 1) \right]$$

$$+ \frac{\eta_2 \cos \eta_2 t}{\eta_2^2 - \phi^2} \left[\sin \phi t \cos \eta_2 t - \frac{\phi}{\eta_2} (\cos \phi t \sin \eta_2 t) \right],$$

$$G_3 = \frac{\phi \sin \eta_1 t}{\eta_1^2 - \phi^2} \left[\cos \eta_1 t \sin \phi t - \frac{\eta_1}{\phi} (\sin \eta_1 t \cos \phi t) \right] +$$

$$\frac{\phi \cos \eta_1 t}{\phi^2 - \eta_1^2} \left[\sin \eta_1 t \sin \phi t - \frac{\eta_1}{\phi} (\cos \eta_1 t \cos \phi t - 1) \right],$$

$$G_4 = \frac{\phi \sin \eta_2 t}{\eta_2^2 - \phi^2} \left[\cos \eta_2 t \sin \phi - \frac{\eta_2}{\phi} (\sin \eta_2 t \cos \phi) \right] + \frac{\phi \cos \eta_2 t}{\phi^2 - \eta_2^2} \left[\sin \eta_2 t \sin \phi - \frac{\eta_2}{\phi} (\cos \eta_2 t \cos \phi - 1) \right] \quad (56)$$

Thus in view of equation (15) taking into account (52) one obtains

$$w(x,t) = \sum_{i=1}^n \left\{ \frac{H_p}{d_1} \frac{[a_1(i,k)\eta_1^2 + a_3(i,k)]}{(\eta_1^2 - \eta_2^2)} (G_1 - G_2) + \frac{H_a}{d_1} \frac{[a_1(i,k)\eta_1^2 + a_3(i,k)]}{(\eta_1^2 - \eta_2^2)} (G_3 - G_4) \right\} \sin \frac{i\pi x}{L} \quad (57)$$

which represents the transverse displacement response of the rotating Timoshenko beam under the action of constant magnitude load. Similarly, in view of expression (16) taking into account of (53) one obtains

$$u(x,t) = \sum_{i=1}^n \left\{ \frac{H_p}{d_1} \frac{a_2(i,k)}{(\eta_1^2 - \eta_2^2)} [G_1 - G_2] + \frac{H_a}{d_1} \frac{a_2(i,k)}{(\eta_1^2 - \eta_2^2)} [G_3 - G_4] \right\} \cos \frac{i\pi x}{L} \quad (58)$$

which is the rotation of the non-prismatic rotating Timoshenko beam under the action of constant magnitude load travelling at time dependent speed

Discussion on the Closed Form Solution

When an undamped system such as this is considered, it is pertinent to examine the resonance condition of the structure. This occurs when the transverse displacement of elastic rotating Timoshenko beam grows without bound. Equation (57) clearly shows that the rotating Timoshenko beam resting on elastic foundation will experience resonance effects whenever

$$\begin{aligned} & \frac{1}{2} (q_a(i,k) \cdot a_3(i,k) + \frac{1}{2} q_b(i,k) \cdot a_1(i,k))^2 \\ & = q_a(i,k) \cdot a_1(i,k) \cdot \{q_b(i,k) \cdot a_3(i,k) - q_c(i,k) \cdot a_2(i,k)\} \\ & \eta_1^2 = \phi^2, \quad \eta_2^2 = \phi^2 \end{aligned} \quad (59)$$

It is also observed that as the foundation moduli and pre-stress increase the critical speed of the dynamical system increases thereby reducing the risk of resonant effects.

Comments on the Numerical Results

For the purpose of Numerical analysis in this study, we consider the initial velocity V_i of the fast-moving concentrated loads to be 8.128m/s and the span L of the beam to be 50m. The value of flexural rigidity γI is 6068242, the values of foundation moduli are varied between 0 N/m³ and 4 x 10⁴ N/ m³, and the values of pre-stress N_f are varied between 0 N/ m³ and 2 x10⁵ N/m³. The results are as shown on the various graphs below.

In figure 1 the transverse displacement response of rotating Timoshenko beams beam under the actions of traveling concentrated loads is displayed. It is clearly seen that when the value of pre-stress N_f is fixed, the displacements of prismatic beam resting on elastic foundation and traversed by concentrated moving loads decreases as the values of foundation modulu E_f increases.

Figure 2 displays the deflection profile of a rotating Timoshenko beam resting on elastic foundation and under the actions of concentrated loads. From the figure it is obvious that as the values of pre-stress N_f increases, for fixed value of foundation moduli E_p , the response amplitudes of the beam decreases.

Figure 3 depicts the deflection profile of the rotating Timoshenko beam resting on elastic foundation and subjected to fast traveling loads. It is shown from the figure that for fixed values of foundation reaction E_p , distance d , load P_i and pre-stress N_p , the deflection of the beam increases as the values of load P_o increases

Similarly, in figure 4 the transverse displacement response of rotating Timoshenko beam subjected to fast traveling loads. The figure shows that for fixed values of

foundation reaction E_f , distance d , load P_0 and pre-stress N_p , the deflection of the beam increases as the values of load P_1 increases Figure 5 displays the response amplitude of

beam subjected to the actions of two moving loads P_1 and P_0 for various values of distance d apart. The response amplitude of decreases as the distance d increases.

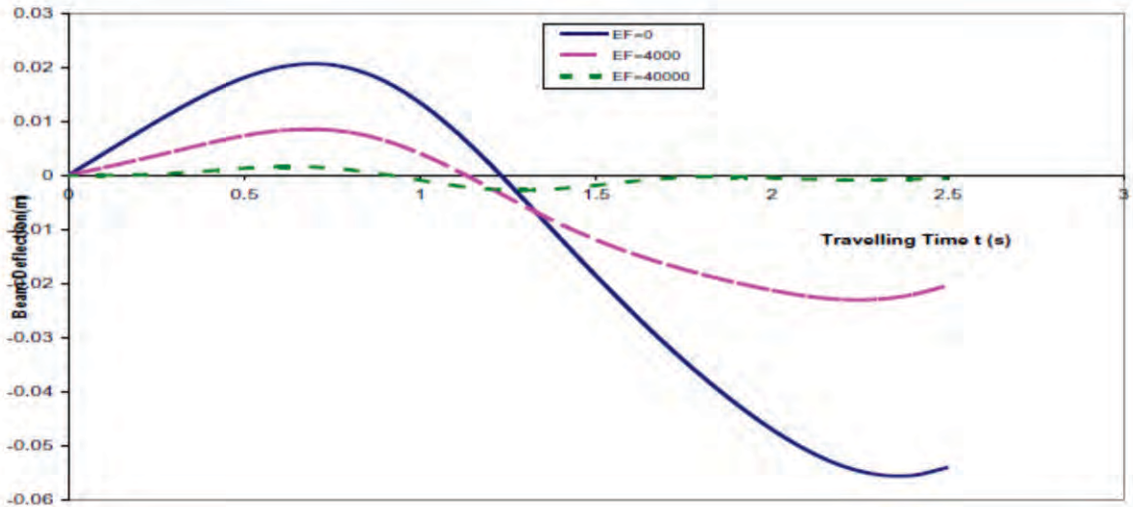


Figure 1: Transverse displacement of a rotating Timoshenko beam under the actions of two moving loads for various values of foundation moduli E_f .

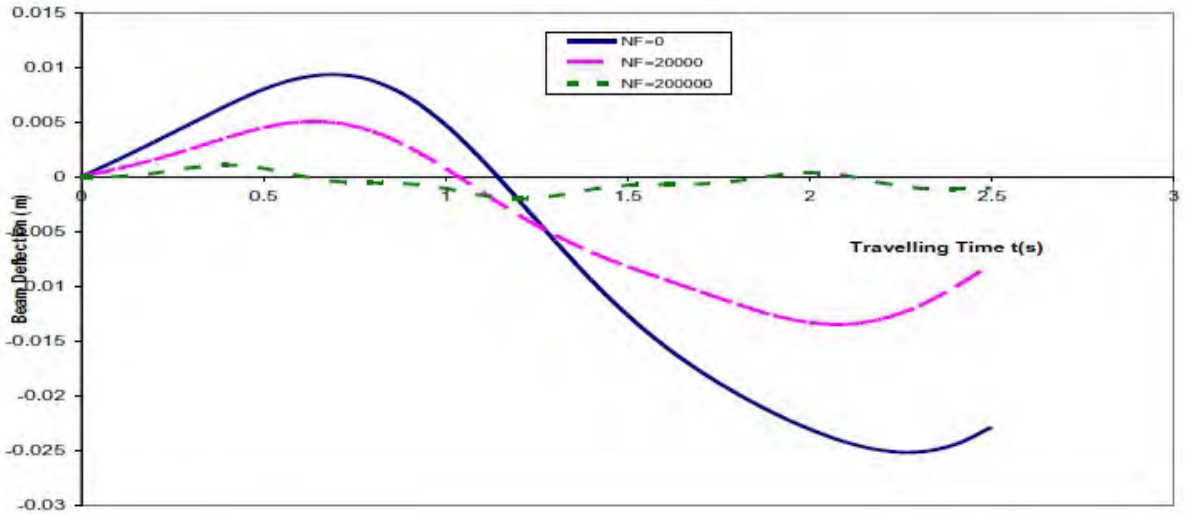


Figure 2: Displacement response of a rotating Timoshenko beam on elastic foundation and traversed by two moving loads for various values of N_p .

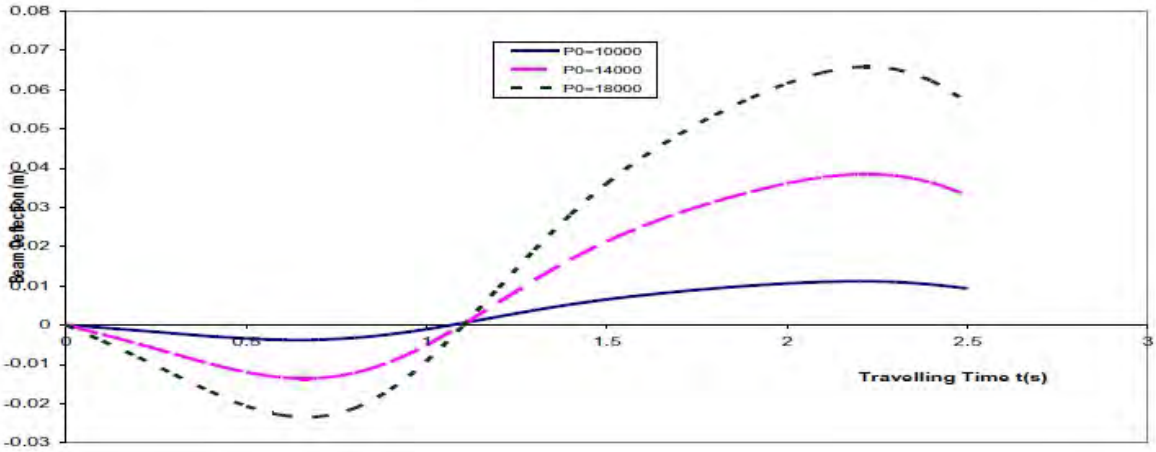


Figure 3: Deflection profile of a rotating Timoshenko beam subjected to two moving loads for various values of load P_0 and for fixed values of prestress N_p , load P_1 , foundation stiffness E_f , distance d

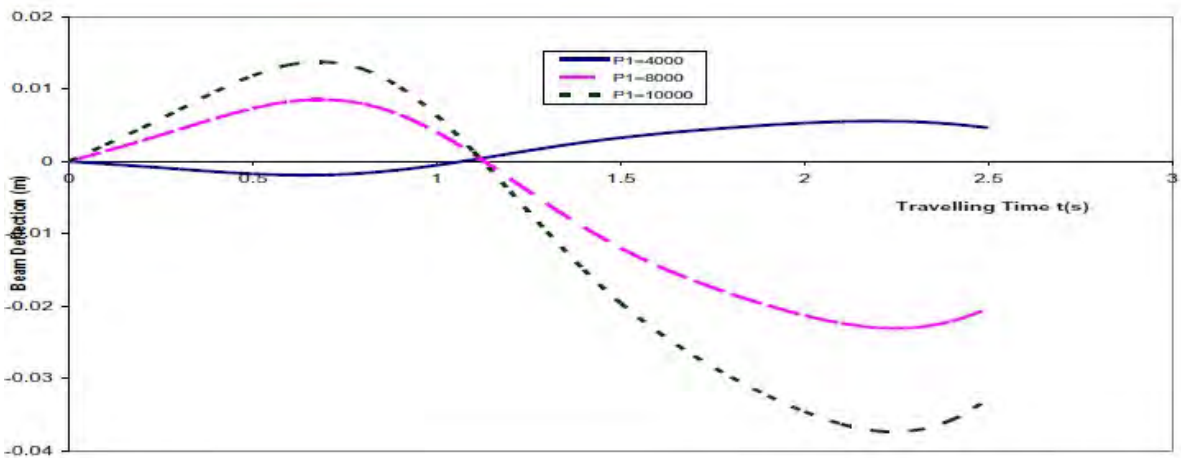


Figure 4: Transverse displacement response of rotating Timoshenko beam resting on elastic foundation and subjected to two moving loads for various values of P_1 and for fixed value of prestress N_p , load P_0 , foundation stiffness E_f , distance d

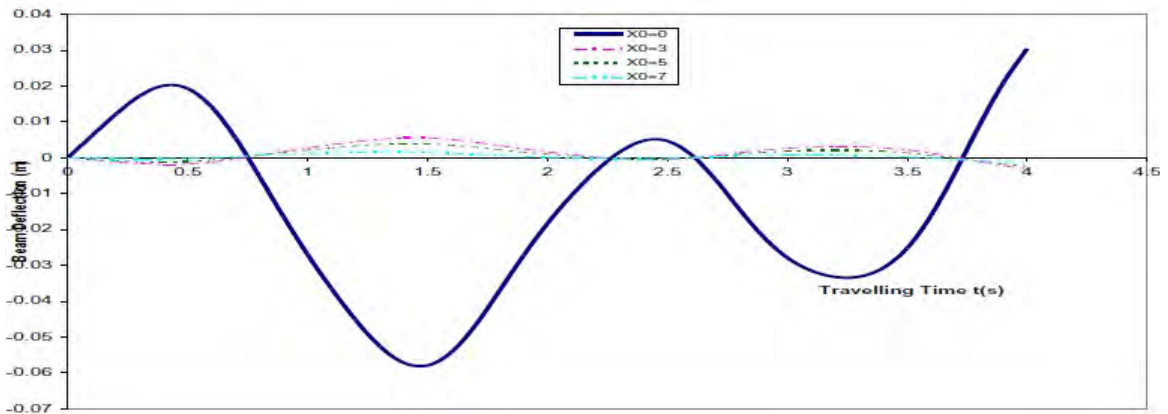


Figure 5. The response amplitude of a Timoshenko beam resting on elastic foundation and under the actions of two moving loads for various values of distance d .

Conclusion

In this work, the dynamical characteristic of the system is investigated. Analytical solution in series form is obtained for the deflection and the rotation of the rotating Timoshenko beam and the effects of foundation stiffness E_f , distance d between the two loads, (P_f and P_o) and the prestress N_f on the vibrating system are investigated. Analytical solution and Numerical result in plotted curves show that;

- (a) as the value of foundation stiffness E_f increases the deflection profile of the rotating Timoshenko beam resting on elastic foundation decreases.
- (b) the dynamic response amplitudes of beam decreases as the distance d between the loads increases when the values of the beam parameters, prestress N_f and foundation modulus E_f are fixed.
- (c) It is also observed that the response amplitudes of the dynamical systems decrease with an increase in the values of prestress N_f .

References

- Adedowole A. (2016). "Flexural motions under moving distributed masses of Beam- type structures on Vlasor foundation and having time dependent boundary conditions." Ph.D thesis. Federal University of Technology, Akure, Nigeria.
- Adedowole, A. (2018). On the response of non-prismatic rotating Timoshenko beam under the actions of concentrated loads travelling at time dependent speeds. *Iconic Res Engr J* **3**(3):102-114.
- Adedowole, A. and Famuwagun, K. S (2017). simply supported non prismatic beam resting on variable elastic foundation carrying travelling harmonic loads. *J Emerging Trends Engr Appl Sci. (JETEAS)* **8**(2): 98-103.
- Adedowole, A. and Jimoh, S.A., (2018): Dynamics analysis of a damped non uniform beam subjected to loads moving with variable velocity. *Archives Cur Res Inter.* **13**(2): 1-16, Article no. ACRI.35919.
- Adeoye, A.S. and Awodola, T. O. (2018). Dynamic Response to Moving Distributed Masses of Pre-stressed Uniform Rayleigh Beam Resting on Variable Elastic Pasternak Foundation. *Edelweiss Appl Sci Techn, (2)*: 1-9,
- Djondjorov P. A. (1995). Invariant properties of Timoshenko beam equations. *J Theoretical Appl Mechanics.* **33**(4):2103-2114.
- Fryba, L. (1972). Vibration of solids and structures under moving loads, Noordhoff International Publishing Groningen, The Netherlands.
- Jimoh S.A. (2017). Analysis of non-uniformly prestressed tapered beams with exponentially varying thickness resting on Vlasov foundation under variable harmonic load moving with constant velocity. *Inter J Advanced Res Publications.* **1**(5):135-142.
- Jimoh, S. A., Ogunbamike, O. K. and Olanipekun, A. O. (2018). Dynamic Response of Non-uniformly Prestressed Thick Beam under Distributed Moving Load Travelling at Varying Velocity. *Asian Res J Math.* **9**(4): 1-18, Article no.ARJOM.41327
- Krylov, A. N. (1905). Mathematical collection of papers of the academy of sciences. Piter burg. Vol. 61.
- Mahmoud A.A., Abdelghany S.M. and Ewis K.M. (2013). Free vibrations of uniform and non- uniform Euler beam using differential

- transformation method. *Asian J Math Applications*, article id ama0097:1-16.
- Meera Saheb., Rao K.G.V. and Janardhan G.R. (2007). Free vibration analysis of Timoshenko beams using coupled displacement field method. *J Structural Engr*, **34**:233-236.
- Ogunbamike, O.K. (2012). Response of Timoshenko beams on Winkler foundation subjected to dynamic load. *Inter J Sci Techn Res.* **1**(8):48-52.
- Omolofe, B., and Adedowole, A. (2017). Response characteristics of non-uniform beam with time-dependent boundary conditions and under the actions of travelling distributed masses, *J Appl Math Computational Mechanics*, **16**(2): 77-99
- Oni, S. T. and Adedowole, A. (2008). Influence of prestress on the response to moving loads of rectangular plates incorporating rotatory inertia correction factor. *J Nigerian Ass Math physics*, **13**: 127-140.
- Oni, S. T. and Omolofe, B. (2005). Dynamic behaviour of Non-Uniform Bernoulli-Euler Beams subjected to concentrated loads travelling at varying velocities. *Abacus, J Math Ass Nigeria*, **32**, no2A,
- Shahin, N. A. and Mbakisya, O., (2010). Simply Supported Beam Response on Elastic Foundation Carrying Repeated Rolling Concentrated Loads. *J Engr Sci Tech.* **5**(1): 52 – 66.
- Stanisic, M. M. Euler, J. A. and Montgomeny, S. T. (1974). On a theory concerning the dynamical behaviour of structures carrying masses. *Ing. Archiv* **43**: 295-305.