www.coast.oaustech.edu.ng
COAST
JOURNAL OF THE SCHOOL OF SCIENCE
Volume 7, Issue 1. May, 2025 ISSN: 2714-3716
Coast, J. Sch. Sci. 7 (1): 1189 - 1199

A REVIEW OF CONTINUOUS UNIVARIATE PROBABILITY DISTRIBUTIONS FROM PEARSON'S FAMILY OF DISTRIBUTIONS TO THE COMPOSITE FAMILY OF DISTRIBUTIONS

***[1]Mazona, V., [2]Adebola, F. B., [3]Akomolafe, A. A. and [4]Ojo, O. O.**

[1,2,3,4]Department of Statistics, Federal University of Technology, Akure, Nigeria.
Corresponding Author's Email: vmmazona@gmail.com

https://dx.doi.org/10.4314/coast.v7i1.7

Abstract

Seeking the flexibility of probability distributions in modeling real life phenomena could be adjudged as the driving force behind the development of new probability distributions. Methods for generating families of continuous univariate probability distributions have received widespread attention in recent decades. It is well established that much of practical statistical studies and developments in probability theory have been dominated by the normal distribution for several decades. However, in the late 19th century, the increasing collection, tabulation, and publication of data by government, private institutions and agencies in demography, social sciences, biology and insurance revealed that the normal distribution was not sufficient for describing phenomena (homogenous with respect to all but random factors) in the real world situations. This reality spearheaded the need to develop other families of distributions that can be well adapted to real-life problems. In this paper, a review of families of continuous univariate probability distributions from the Pearson's family down to the composite family of distributions.

Keywords: continuous distribution; cumulative distribution function; probability density function; univariate distributions

Mathematics Subject Classification: 60G70; 62E20; 62F35; 62P05; 68W40

Introduction

The development of new flexible and highly adaptive univariate probability distributions has been at the forefront of the development of statistical theory. Beginning from the phenomenal work of Pearson (1895) down to the recent works of the contemporary scholars. The theory of continuous univariate probability distributions has become one of the most widely studied theories within the vanguard of statistical modeling. Pearson (1895) introduced a system of differential equation from which continuous probability density functions (pdfs) can result as solutions. Owing to the difficulties in finding analytic solutions to the Pearson's system, many other methods for generating either probability density function (pdfs) or cumulative distribution functions (cdfs) have appeared in the literature. Indeed, the development of new continuous univariate probability distributions is based on one fundamental premise – the need to arrive at models capable of fitting real life data adequately. The classical continuous univariate probability distributions, which contain one or two parameters, have been observed to break down when complexities exist in data structure. For example, the Weibull distribution was defined to handle random processes that are monotone in nature. When the monotonicity property is not present in data arising from real-life processes, it becomes inadequate to use the Weibull distribution to capture the behavior of such non-monotone processes. This situation has led to the formulation of various generalizations of the Weibull distribution so that it can handle both monotone and non-monotone processes.

Generally, the development of new continuous univariate probability distributions has been carried out either by adding extra parameter(s) to an already existing continuous univariate distribution or by combining two or more probability distributions to obtain new ones (Lee et al., 2013). In this paper, we perform a review on the development of continuous univariate distributions from Pearson's family to the composite family of univariate distributions.

In Section 2, we looked at the families of continuous univariate distributions developed before the composite families of distributions. In Section 3, we examine the composite family of distributions. The paper closes in Section 4 with a conclusion.

Families of continuous univariate distributions before the composite families of distribution

We shall consider the Pearson family of distributions, the skewed family of distributions, the exponentiated family of distributions and the beta-generated family of distributions.

(a) The Pearson's family

The oldest family of probability distribution is the Pearson family of distributions which was developed by Pearson (1895) and it was based on the solution of the differential equation

$$\frac{1}{f(x)}\frac{df(x)}{dx}=\frac{a+x}{c_0+c_1x+c_2x^2} \qquad (1)$$

where a, c_0, c_1 and c_2 are the parameters which control the shape of the distribution. The roots of the equation $c_0 + c_1x + c_2x^2 = 0$ determine the form of solution of (1). In particular, when $c_1 = c_2 = 0$, the solution of (1) results to the normal distribution. Detailed discussion on the various types of Pearson classification is presented in Johnson *et al.* (1994). An extension of the Pearson's system was carried out by Dunning and Hanson (1977) and since then several works have appeared in the literature by adopting the framework. For example, the works Burr (1942), Fry (1993), Johnson *et al.* (1994), Asadl (1998), Nadarajah and Kotz (2007) and Kibria and Shakil (2011). The major advantage of the Pearson's families of distributions is that it allow one to generate non-symmetric distribution and this was an improvement on the previously held positions which makes the generation of non-symmetric distributions impossible. However, the drawback of the Pearson's family lies in the fact that not too many analytic solutions can result from the system in (1) and also, no probability distribution generated from (1) is suitable in fitting data sets with multi-component structure.

(a) The skewed family

The novel work on the method for generating skewed distributions based on the combination of two symmetric distributions is due to Azzalini (1985). The Azzalini framework starts off with a random variable X with a pdf symmetric about zero and another symmetric random variable γ with absolutely continuous cdf G (x) and a pdf G' (x). Then, for any real number λϵR,

$$P(Y - \lambda X < 0) = E_X[P(Y < \lambda x | X = x)] = \int_{-\infty}^{\infty} f_0(x) G(\lambda x) dy = 1/2.$$

Thus,

$$2f_0(x)G(\lambda x), \qquad -\infty < x < \infty, \qquad (2)$$

is a valid pdf. The equation (2) provides a general framework for generating skewed distributions for any choice of G(x) and F_0(x) with λ as the skew parameter. The survey paper by Kotz and Vicari (2005) gave a comprehensive summary of the development, characterizations and generalizations of the family of distribution in (2). A summary of the list of properties of the family of distribution was presented in Lee *et al.* (2013). Other generalizations of (2) are found in Azzalini (1986), Arnold *et al.* (1993), Pewsey (2000), Balakrishnan (2002), Arellano-Valle et al. (2004), Azzalini (2005), and Choudhury and Matin (2011). While the skew family of distributions provides us with a general framework for generating skewed distributions which are very important for applications, it is limited by the fact that distributions emanating from the framework are still unsuitable for fitting multi-component data sets.

(C) The exponentiated family

The exponentiated family of distributions is due to Mudholkar and Srivastava (1993). The approach is based on a random variable X having a baseline cdf F(x) and a parameter vector θ Mudholkar and Srivastava (1993) define a new family of distributions by the cdf of the form

$$G(x) = [F(x)]^{\theta}, x \in \mathbb{R}, \theta > 0, \qquad (3)$$

where for θ = 1, (3) reduces to the cdf of the baseline distribution. Thus, the parameter θ is an added parameter to the baseline distribution F(x). Several families of exponentiated distributions have been proposed in the literature. Some of them are found in the works of Gupta and Kundu (1999), Kundu and Raqab (2005), Nadarajah (2005), Elbatal (2011), Nadarajah et al., (2011), Abdul-Moniem and Abdel-Hammed (2012), El-Gohary et al., (2013), Ashour and Eltehiwy (2015) and Chaubey and Zhang (2015). While it is possible to generalize any probability distribution using the system in (3), members of the exponentiated family of distributions are however not suitable for fitting data with multi-components.

(d) The beta-generated family

The beta-generated family of distribution is due to Eugene et al. (2002). The cdf of a beta-class random variable X is defined as

$$G(x) = \int_0^{F(x)} \frac{1}{B(\alpha, \beta)} t^{\alpha-1}(1-t)^{\beta-1} dt, \ x \epsilon \mathbb{R}, \alpha, \beta > 0, \qquad (4)$$

Where B(.,.) is the complete beta function and F(x) is the cdf of any continuous random variable X. Several beta-generated probability distributions have appeared in the literature by taking F(x) as the cdf of different distributions. Some examples are found in the works of Nadarajah and Kotz (2004), Famoye et al., (2005), Nadarajah and Kotz (2006), Akinsete et al., (2008), Nadarajah and Gupta (2004), Kong et al., (2007), Pescim et al., (2010), Condino et al., (2010), Alshawarbeh et al., (2012) and MirMostafaee et al., (2015). The beta-generated family presents the largest class of probability distributions, but it is still not suitable for fitting heterogeneous data sets.

The composite families of distribution

Here we shall look at the various composite families of distributions as proposed in the literature for handling multi-component data such as composite framework by Cooray and Ananda (2005), Scollnik composite family of Distributions, Half-range composite family of distributions, the heavy-tailed two-component composite family of distributions. The Cooray and Ananda composite family of distribution is used to start the subsection.

(a) The Cooray and Ananda composite family of distributions

The composite family of distribution was developed by Cooray and Ananda (2005) within the context of application to insurance data. The composite method involved the generation of distribution functions that can be used to model multi-component data. Unlike the previous families which precede the composite method, only the composite method can be used to generate distributions that can handle multi-component data. The composite method of Cooray and Ananda (2005) involves combining two distributions with densities f_1 and f_2. A random variable X is said to belong to a composite family of distributions if its pdf f(x) can be written as

$$f(x)=\begin{cases} af_1(x),\ if\ x\epsilon(-\infty,\ u] \\ af_2(x),\ if\ \ x\epsilon[u,\infty), \end{cases} \tag{5}$$

where α is the normalizing constant, f_1 and f_2 are pdfs with support on the real line. The unknown μ is determined so that the newly formed pdf f(x) is continuous and differentiable at μ. The continuity and differentiability conditions are achieved by imposing the following constraints:

$$f_1(u)=f_2(u)\ and f_1'(u)=f_2'(u).$$

Various composite distributions have been formulated in the literature and examples include: the composite log normal-Pareto distribution by Cooray and Ananda (2005), the composite Inverse gamma-Pareto distribution by Aminzadeh and Deng (2019), the Weibull and inverse Weibull composite distribution by Cooray et al. (2010) and Composite Weibull-Pareto distribution by Cooray (2009).

Taking $f_1(x)$ and $f_2(x)$ and as the pdf of the lognormal and Pareto distribution respectively, Cooray and Ananda (2005) used the system in (5) to define the composite lognormal Pareto distribution with pdf expressed as

$$f(x)=\begin{cases} \dfrac{\alpha u^{\alpha}}{(1+\Phi(z))x^{\alpha+1}}\exp\left[-\dfrac{\alpha^2}{2z^2}\ln(x/u)\right] & if\ x\epsilon(0,\ u] \\ \dfrac{\alpha u^{\alpha}}{(1+\Phi(z))x^{\alpha+1}}, & if\ x\epsilon[u,\infty), \end{cases} \tag{6}$$

Where Φ (Z) is the cumulative distribution function of the standard normal distribution, and Z is a known constant given by the positive solution of the equation exp $(-z^2)$ = $2\varpi k^2$. The authors consequently applied the model in fitting insurance and financial data.

Preda and Ciumara (2006) also used the system in (5) to define the composite Weibull Pareto distribution by taking $f_1(x)$ and $f_2(x)$ as the pdf of the Weibull and Pareto distribution respectively. The pdf of the composite Weibull Pareto distribution was given by

$$f(x)=\begin{cases} \dfrac{(k+1)^2\alpha}{(2k+1)x}\left(\dfrac{x}{u}\right)^{\alpha k}\exp\left[-\left(\dfrac{k+1}{k}\right)\left(\dfrac{x}{u}\right)^{\alpha k}\right] & if\ x\epsilon(0,\ u] \\ \left(\dfrac{k+1}{2k+1}\right)\left(\dfrac{\alpha}{x}\right)\left(\dfrac{u}{x}\right)^{\alpha}, & if\ x\epsilon[u,\infty), \end{cases}, \tag{7}$$

where the value of K is a known constant given by the positive solution of the equation

$$\exp\left(1+\frac{1}{k}\right)=k+1.$$

The authors provided expressions for the moments of the distribution as well as the likelihood function and consequently established a maximum likelihood estimation procedure for the distribution. The distribution was further tested using simulated data sets.

Benatmane et al. (2020) also used the system in (5) to generate the composite Rayleigh Pareto distribution by taking $f_1(x)$ and $f_2(x)$ as the pdf of the Rayleigh and Pareto distribution respectively to define a pdf with the form

$$f(x)=\begin{cases}1.551\dfrac{x}{u}\exp\left[-1.35\dfrac{x^2}{u^2}\right] & if\ x\epsilon(0,u]\\ \dfrac{0.40209}{x^{1.69996}}u^{0.69995}, & if\ x\epsilon[u,\infty),\end{cases}\quad (8)$$

The authors observed that the density has a similar shape as the lognormal Pareto distribution although with larger tail. The distribution was further applied in fitting simulated data and two real insurance data namely the Algerian fire insurance and the Danish fire insurance data. Other works developed using (5) include the composite exponential-Pareto distribution (Teodorescu and Vernice, 2006, 2009).

(a) Scollnik composite family of distributions

Scollnik (2007) offered a different variant of composite family of distribution different from that of Cooray and Ananda (2005) by allowing for flexibility of the mixing weights thus replacing the constant weights as found in the framework of Cooray and Ananda (2005). This approach was further extended by Scollnik and Sun (2012). The Scollnik and Sun (2012) model is of the form

$$f(x)=\begin{cases}rf_1^*(x), & if\ x\epsilon(-\infty,u]\\ (1-r)f_2^*(x), & if\ x\epsilon[u,\infty),\end{cases}\quad (9)$$

where

$$f_1^*(x)=\frac{f_1(x)}{f_1(u)}\ \text{and}\ f_2^*(x)=\frac{f_2(x)}{1-F_2(u)},$$

$f_i(.)$ and $F_i(.)$ are pdf and cdf of some distributions respectively and r is the mixing weight. Scollnik and Sun (2012) took $f_1(x)$ and $f_2(x)$ to be the Weibull and the Lomax distributions respectively with pdfs

$$f_1(x)=\frac{\alpha}{\lambda}\left(\frac{x}{\lambda}\right)^{\alpha-1}\exp\left[-\left(\frac{x}{\lambda}\right)^{\alpha-1}\right], x>0,$$

$$f_2(x)=\frac{\beta\sigma^{\beta}}{(\sigma+x)^{\beta+1}}, x>0.$$

Applying the continuity condition that f(x) is right continuous and left continuous at the threshold μ, Nadarajah and Baker (2014) obtained the value of r as

$$r=\frac{f_1(u)[1-F_2(u)]}{f_1(u)[1-F_2(u)]+f_2(u)F_1(u)},$$

Where $0\le r\le 1$.

Teodorescu and Vernice (2013) studied some mathematical properties of the families of Composite distributions in (9). They obtained the cdf of the density in (9) as

$$F(x)=\begin{cases}r\dfrac{F_1(x)}{F_1(u)}, & if\ x\epsilon(-\infty,u]\\ r+(1-r)\dfrac{F_2(x)-F_2(u)}{1-F_2(u)}, & if\ x\epsilon[u,\infty).\end{cases}$$

They also obtained the n-th order initial moment of the random variable following the density in (9), $E_n(f)$ as

$$E_n(f)=rE_n(f_1^*)+(1-r)E_n(f_2^*),$$

where

$E_n(.)=E(X^n)$

The characteristic function was also obtained as

$$\varphi_f(t)=r\varphi_{f_1^*}(t)+(1-r)\varphi_{f_2^*}(t),\qquad t\epsilon R,$$

where

$\varphi(.)(t)=E(e^{itx})$

Teodorescu and Vernice (2013) also offered a framework for generating random samples from the distribution as well as a methodology for the estimation of the parameters of the distribution.

(a) Half-range composite family of distributions

The family of composite distributions defined by Cooray and Ananda (2005) has its range or support in the entire real line. Baker et al. (2015) arguing that since loss data are assume positive values, there is need to half the real line and in the process they developed a new families of composite distributions with pdf of the form

$$f(x)=\begin{cases}\dfrac{1}{1+\phi}f_1^*(x), & if\ x\epsilon(0,u]\\ \dfrac{\phi}{1+\phi}f_2^*(x), & if\ x\epsilon[u,\infty).\end{cases}\quad (10)$$

Baker et al. (2015) also imposed the continuity and differentiability condition on the density in (10) in order to make the density smooth. Some other results obtained by the authors include:

$$\phi = \frac{\frac{d}{dp}\ln F_1(u)}{\frac{d}{dp}\ln(1 - F_2(u))},$$

and the solution of the equation

$$\frac{d}{dp}\ln\left[\frac{F_1(u)}{F_2(u)}\right] = 0$$

For μ will give an estimate for μ. Using (10), the authors defined the composite Weibull transformed beta distribution which has the composite Weibull-Burr XII, composite Weibull-log logistic, composite Weibull-inverse paralogistic, composite Weibull-generalized Pareto and composite Weibull-inverse Pareto as sub distributions.

(a) The heavy-tail two-component composite family of distributions

The procedural attempts in the modeling of multi-component data particularly those exhibiting heavy-tailed to the right can be found in some forms in the works of Mandava et al. (2011), Carreau and Bengio (2009), Kuhl and Bhairgond (2000), Debbabi (2015), Li et al (2012) and Kollu et al (2012). However, Debbabi et al. (2016) put this framework into a unified family of hybrid distributions of the form

$$h(x) = \begin{cases} f_N(x), \, if \; x \le u \\ f_{GPD}(x), \, if \quad x > u \end{cases} \tag{11}$$

Where $\underline{F}_N$ and F_{GPD} are the probability density functions of the normal distribution and the generalized Pareto distribution (GPD) respectively and μ is the junction point. The distribution in (11) was adapted to two-component heavy-tailed distributions with tails to the right which is normally exhibited by many data sets. The authors further specified the explicit form of the model as

$$h^{\mu,\sigma}_{\xi,\beta}(x,u) = \begin{cases} \gamma f_{\mu,\sigma}(x), \, if \; x \le u \\ \gamma g_{\xi,\beta}(x-u), \, if \quad x > u, \end{cases} \tag{12}$$

Where μϵR and σ > 0 are the mean and standard deviation of the Gaussian distribution with pdf $F_{\mu,\sigma}$ expressed as

$$f_{\mu,\sigma}(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \forall x \in R,$$

While ξ and ϐ represent the tail index and the scale parameter respectively of the GPD with pdf gξ ϐ expressed as

$$g_{\xi,\beta}(x) = \begin{cases} \frac{1}{\beta}\left(1+\frac{\xi}{\beta}x\right)^{-1-\frac{1}{\xi}} & if \;\; \xi \neq 0 \\ \frac{1}{\beta}\exp\left(-\frac{x}{\beta}\right), & if \;\; \xi \neq 0 \end{cases} \qquad x \epsilon D(\xi,\beta),$$

where

$$D(\xi,\beta) = \begin{cases} [0,\infty), \;\; if \;\; \xi \neq 0 \\ \left[0, -\frac{\xi}{\beta}\right], if \;\; \xi \neq 0 \end{cases} \quad .$$

The parameter γ was defined as a regulating parameter rather than as a weight in order to ensure that $\int_R h^{\mu,\sigma}_{\xi,\beta} = 1$. In order to determine the value of this parameter, the authors further defined the pdf in (12) as

$$h^{\mu,\sigma}_{\xi,\beta}(x,u) = \gamma(1 - H(x-u))f_{\mu,\sigma}(x) + \gamma H(x-u)g_{\xi,\beta}(x), \tag{13}$$

Where H is the Heaviside function defined as

$$H(x) = \begin{cases} 1, & if \; x > 0 \\ 0, & otherwise \end{cases} \quad .$$

Thus by integrating (13) and equating to 1, the authors obtained the value of γ as

$$\gamma = \frac{1}{F_{\mu,\sigma}(u) + 1},$$

where $F_{\mu,\sigma}$ is the cdf of the Gaussian distribution. The authors developed an unsupervised algorithm for the estimation of the free parameters of the distribution and suggested that any other distribution could be used in place of the Gaussian distribution while retaining the GPD at all times.

Though the literature on heavy-tail two-component composite distributions is sparse, Osatohanmwen et al. (2024) recently offered a general framework for generating a three-component composite distribution that included the two-component families. The general three-component framework can be seen as the general family of composite distributions, which includes all other families of composite distributions. Osatohanmwen (2025) also studied a two-component variant of the three-component framework called the half-normal-GPD distribution and applied it to financial and rainfall data.

Conclusion

In this paper, we have reviewed various families of continuous univariate distributions, highlighting their properties, applications, and significance in statistical modeling. These distributions serve as fundamental tools for understanding and analyzing real-world data, each with unique characteristics suited for different types of modeling scenarios. Our discussion covered families of distributions before and after the composite families of distributions. Additionally, we explored more flexible and generalized families that allow for greater adaptability in modeling skewed, heavy-tailed, or multimodal data. Understanding the strengths and limitations of these distributions is crucial in selecting the appropriate model for a given dataset. Future research may focus on developing new families of distributions that better capture complex real-world phenomena, as well as advancing computational methods for parameter estimation and inference. By synthesizing the key features of these distribution families, this review provides a foundation for further exploration and application in fields such as finance, engineering, environmental science, and medical research. The choice of an appropriate univariate continuous distribution remains a critical step in statistical analysis, and continued advancements in this area will enhance the robustness and accuracy of probabilistic modeling.

References

Abdul-Moniem, B.I. and Abdel-Hameed, H.F. (2012). On Exponentiated Lomax Distribution.

Akinsete, A., Famoye, F. and Lee, C. (2008). The beta-Pareto distribution. *Statistics,* 42, 547 – 563.

Alshawarbeh, E., Lee, C. and Famoye, F. (2012). Beta Cauchy distribution. *Journal of Probability and Statistical Science,* 10, 41 – 57.

Aminzadeh, M. S. and Deng, M. (2019). Bayesian predictive modeling for inverse gammapareto composite distribution. Communications in Statistics: Theory and Methods, 48, 1938–1954.

Arellano-Valle, R.B., Gomez H.W. and Quintana, F.A. (2004). A new class of skew-normal distributions. *Communications in Statistics Theory & Methods*, 33, 1465 – 1480.

Arnold, B.C., Beaver, R.J., Groeneveld, R.A. and Meeker W.Q. (1993). The nontruncated marginal of a truncated bivariate normal distribution. *Psychometrika*, 58, 471 – 488.

Asadl, M. (1998). Characterization of the Pearson system of distributions based on reliability measures. *Statistics Papers*, 39, 347 – 360.

Ashour, S.K. and Eltehiwy, M.A. (2015). Exponentiated power Lindley distribution. *Journal of Advanced Research*, 6, 895 – 905.

Azzalini, A. (1985). A class of distributions which includes the normal ones. Scandinavian Journal of Statistics, 12, 171–178.

Azzalini, A. (1986). Further results on a class of distributions which includes the normal ones.

Azzalini. A. (2005). The skew-normal distribution and related multivariate families (with discussion), *Scandinavian Journal of Statistics,* 32, 159 – 188.

Bakar, S. A. A., Hamzah, N. A., and Nadarajah, S. (2015). Modeling loss data using composite models. Insurance: Mathematics and Economics, 61, 146–154.

Balakrishnan, N. (2002). Discussion of skewed multivariate models related to hidden truncation and/or selective reporting by Arnold and Beaver. *Test,* 11, 37 - 39.

Benatmane, C., Zeghdoudi, H., Shanker, R., and Lazri, N. (2020). Composite Rayleigh-Pareto distribution: Application to real fire insurance losses data set. Journal of Statistics and Management Systems, DOI:10.1080/09720510.2020.1759253 pp. 1–13.

Burr, I.W. (1942). Cumulative frequency functions. *Annals of Mathematical Statistics*, 13, 215 - 232.

Carreau, J. and Bengio, Y. (2009). A hybrid Pareto Mixture for conditional assymetric fattailed distributions. IEEE Transactions on Neural Networks, 7, pp. 1087–1101.

Chaubey, Y.P. and Zhang, R. (2015). An extension of the Chen's family of survival distributions with bathtub shape or increasing hazard rate function. *Communications in Statistics – Theory & Method*, 44, 4049 – 4064.

Choudhury K. and Matin, M.A. (2011). Extended skewed generalized normal distribution. *Metron*, LXIX, 265 – 278.

Communications in Statistics – Theory and Methods, 31, 497–512.

Condino, F. and Domma, F. (2010). The beta Dagum distribution. Presented at the *45th Scientific Meeting of the Italian Statistical Society*, University of Padua, June 15 – June 18.

Cooray, K. (2009). The Weibull–Pareto Composite Family with Applications to the Analysis of Unimodal Failure Rate Data. Communications in Statistics: Theory and Methods, 38, 1901– 1915.

Cooray, K. and Ananda, M. (2005). Modeling actuarial data with a composite lognormal –Pareto model, *Scandinavian Actuarial Journal*, 5, 321 – 334.

Cooray, K. and Ananda, M. M. A. (2005). Modeling actuarial data with a composite lognormalPareto model. Scandinavian Actuarial Journal, pp. 321–334.

Cooray, K., Gunasekera, S., and Ananda, M. (2010). Weibull and inverse Weibull composite distribution for modeling reliability data. Model Assisted Statistics and Applications, 5(2), 109–115.

Debbabi, N., El Asmi, S., and Mboup, M. (2015). Distribution hybride pour la mod elisat6ion de donn ees a deux queues lourdes: Application sur les donn eesneuronales. In Grouped'Etudes du Traitement du Signal et des Images,GRETSI.

Debbabi, N., El Asmi, S., and Mboup, M. (2015).Distribution hybride pour la mod_elisation de donn_ees _a deux queues lourdes: Application sur les donn_ees neuronales. In Groupe d'Etudes du Traitement du Signal et des Images,GRETSI

Debbabi, N., Kratz, M., and Mboup, M. (2016). A Self-Calibrating Method for Heavy Tailed

Debbabi, N., Kratz, M., and Mboup, M. (2016). A Self-Calibrating Method for Heavy Tailed Modeling: Application in Neuroscience and Finance. ESSEC working paper 1619.

Dunning, K. and Hanson, J.N. (1977). Generalized Pearson system of distributions and linear programming. *Journal Statistics and Computer Simulations,* 6, 115 - 128.

Elbatal, I. (2011). Exponentiated modified Weibull distribution. *Economic Quality Control.* 26, 189 – 200.

El-Gohary, A., Alshamrani, A. and Al-Otaibi, A.N. (2013). The generalized Gompertz distribution. *Applied Mathematical Modelling,* 37, 13 - 24.

Eugene, N., Lee, C., and Famoye, F. (2002). Beta-normal distribution and its applications.

Famoye, F., Lee, C. and Olumolade, O. (2005). The beta-Weibull distribution, *Journal of Statistical Theory and Applications*, 4, 121 - 136.

Fry, T.R.L. (1993). Univariate and Multivariate Burr distributions: A survey. *Pakistan Journal of Statistics Series A*, 9, 1 – 25.

Gupta, R.D. and Kundu, D. (1999). Generalized exponential distributions. *Australian and New Zealand Journal of Statistics*. 41, 173 – 188.

In Simulation Conference Proceedings Winter, 2000, vol. 1, pp. 562–571.

International Journal of Mathematical Archive, 3, 2144 - 2150.

Johnson, N.L., Kotz, S. and Balakrishnan, N. (1994). *Continuous univariate distributions*, vol. 1, second edition, New York, John Wiley and sons, Inc.

JooSeuk, K., and Clayton, D. (2012). Robust kernel density estimation. *Journal of Machine Learning Research*, 13, 2529-2565.

Kibria, B.M.G. and Shakil, M. (2011). A new five-parameter Burr system of distributions based on generalized Pearson differential equation. *Proceedings, Section on Physical and Engineering Sciences,* Joint Statistical Meeting, Baltimore.

Kollu, R., Rayapudi, S. R., Narasimham, S., and Pakkurthi, K. M. 1 (2012). Mixture probability distribution functions to model wind speed distributions*. International Journal of Energy and Environmental Engineering* 3, 1-10.

Kollu, R., Rayapudi, S. R., Narasimham, S., and Pakkurthi, K. M. (2012). Mixture probability distribution functions to model wind speed distributions. International Journal of Energy and Environmental Engineering, 3, 1–10.

Kong, L., Lee, C. and Sepanski, J.H. (2007). On the properties of beta-gamma distribution, *Journal of Modern Applied Statistical Methods*, 6, 187 - 211.

Kotz, S. and Vicari, D. (2005). Survey of developments in the theory of continuous skewed distributions. *Metron,* LXIII, 225 - 261.

Kuhl, M. E. and Bhairgond, P. S. (2000). Non-homogeneous Poisson processes using wavelets.

Kundu, D. and Raqab. M. Z. (2005). Generalized Rayleigh distribution: Different methods of estimation. *Computational Statistics and Data Analysis*, 49, 187 - 200.

Lee, C., Famoye, F., and Alzaatreh, A. (2013). Methods for generating families of univariate continuous distributions in the recent decades. WIREs Comp Stat, 5, 219–238.

Li, C., Singh, V. P., and Mishra, A. K. (2012). Simulation of the entire range of daily precipitation using a hybrid probability distribution. *Water resources research,* 48, 1-17.

Li, C., Singh, V. P., and Mishra, A. K. (2012). Simulation of the entire range of daily precipitation using a hybrid probability distribution. Water resources research, 48, 1–17.

Mandava, A., Latifi, S., and Emma, R. (2011). Reliability assessment of microarray data using fuzzy classification methods: A comparative study. Communications in Computer and Information Science, 190, 351–360.

MirMostafaee, S.M.T.K., Mahdizadeh, M. and Nadarajah, S. (2015). The beta Lindley distribution. *Journal of Data Science,* 13, 603 - 626.

Modeling: Application in Neuroscience and Finance. *ESSEC working paper* 1619, 1 – 26.

Mudholkar, G. S. and Srivastava, D. K. (1993). Exponentiated Weibull family for analyzing bathtub failure-rate data. IEEE Transactions on Reliability, 42, 299–302.

Nadarajah, S. (2005). Exponentiated Pareto Distributions. *Statistics*, 39, 255 - 260.

Nadarajah, S. and Bakar, S. (2014). New composite models for the Danish fire insurance data.

Nadarajah, S. and Gupta, A.K. (2004). The beta-Frechet distribution. *Far East Journal of Theoretical Statistics*, 14, 15 – 24.

Nadarajah, S. and Kotz, N. (2007). Moments of a product Pearson type VII density distribution. *AStA Advance Statistics Papers*, 91, 441 – 448.

Nadarajah, S. and Kotz, S. (2004). The beta-Gumbel distribution, *Mathematical Problems in Engineering*, 4, 323-332.

Nadarajah, S. and Kotz, S. (2006). The beta exponential distribution, *Reliability Engineering and System Safety*, 91, 689 - 697.

Nadarajah, S. and Kotz, S. (2006). The beta exponential distribution, *Reliability Engineering and System Safety*, 91, 689 - 697.

Nadarajah, S. Bakouch, H.S., and Tahmasbi, R.A. (2011). Generalized Lindley distribution.

Orbanz, P., and Teh, Y. W. (2010). Bayesian Nonparametric Models. *Springer US*, pp. 81- 89.

Osatohanmwen, P. (2025). A new two-component hybrid model for highly right-skewed data: estimation algorithm and application to finance and rainfall data. Bozen Economics & Management Paper Series, NO 108/2025 https://ideas.repec.org/s/bzn/wpaper.html

Osatohanmwen, P., Oyegue, F.O., Ogbonmwan, S.M. and William, Muhwava. (2024). A General Framework for Generating Three-Components Heavy-Tailed Distributions with Application. Journal of Statistical Theory and Applications. 2024(23), 290-314.

Parzen, E. (1962). On estimation of a probability density function and mode. *The annals of mathematical statistics*, 33, 3, 1065-1076.

Pearson, K. (1895). Contributions to the mathematical theory of evolution, II. Skew variation in homogenous material. Philosophical Transactions of the Royal Society London A186, 343–414. Pescim, R.R., Demetrio, C.B.G., Cordeiro, G.M., Ortega, E.M.M. and Urbano, M.R. (2010). The beta generalized half-normal distribution. *Computational Statistics and Data Analysis*, 54, 945 – 957.

Pewsey, A. (2000). The wrapped skewed-normal distribution on the circle. *Communications in Statistics - Theory & Method,* 29: 2459 – 2472.

Preda, V. and Ciumara, R. (2006). On composite models: Weibull-Pareto and Lognormal- Pareto: A comparative study. Rom. J. Econ. Forecast, 3: 32–46.

Sankhya B, 73, 331 – 359.

Scand. Actuarial Journal, 2014(2), 180–187.

Scollnik, D. P. (2007). On composite Lognormal-Pareto models. Scandinavian Actuarial Journal, 1: 20–33.

Scollnik, D. P. and Sun, C. (2012). Modeling with Weibull-Pareto models. North American Actuarial. Journal, 16(2): 260–272.

Series B (Statistical Methodology), 61, 485-527.

Statistica, 46, 199 – 208.

Teodorescu, S. and Vernice, R. (2006). A composite exponential-Pareto distribution. An St UnivOvidius Constanta, 14: 99–108.

Teodorescu, S. and Vernice, R. (2009). Some composite exponential-Pareto models for Actuarial Prediction. Romanian Journal of Economic Forecast, 12, 82–100.

Teodorescu, S. and Vernice, R. (2013). On some Pareto Models. Math. Reports, 1, pp. 11–29.

Walker, S. G., Damien, P., Laud, P. W., and Smith, A. F. M. (1999). Bayesian nonparametric inference for random distributions and related functions. *Journal of the Royal Statistical Society:*